

BOTT PERIODICITY FOR FIBRED CUSP OPERATORS

FRÉDÉRIC ROCHON

ABSTRACT. In the framework of fibred cusp operators on a manifold X associated to a boundary fibration $\Phi : \partial X \rightarrow Y$, the homotopy groups of the space $G_{\Phi}^{-\infty}(X; E)$ of invertible smoothing perturbations of the identity are computed in terms of the K -theory of T^*Y . It is shown that there is a periodicity, namely the odd and the even homotopy groups are isomorphic among themselves. To obtain this result, one of the important steps is the description of the index of a Fredholm smoothing perturbation of the identity in terms of an associated K -class in $K_c^0(T^*Y)$.

INTRODUCTION

For standard pseudodifferential operators on a closed manifold X acting on some complex vector bundle E , Bott periodicity arises by considering the group

$$G^{-\infty}(X; E) = \{\text{Id} + Q \mid Q \in \Psi^{-\infty}(X; E), \text{Id} + Q \text{ is invertible}\}$$

of invertible smoothing perturbations of the identity. This becomes a topological group by taking the \mathcal{C}^∞ -topology induced by the identification of smoothing operators with their Schwartz kernels, which are smooth sections of some bundle over $X \times X$. If Δ^E is any Laplacian acting on sections of E , and if $\{f_i\}_{i \in \mathbb{N}}$ is a basis of $L^2(X; E)$ coming from a sequence of orthonormal eigensections of Δ^E with increasing eigenvalues, then there is an isomorphism of topological groups

$$\begin{aligned} f_{\Delta^E} : G^{-\infty}(X; E) &\rightarrow \mathcal{G}^{-\infty} \\ \text{Id} + Q &\mapsto \delta_{ij} + \langle f_i, Q f_j \rangle, \end{aligned}$$

where $\mathcal{G}^{-\infty}$ is the group of invertible semi-infinite matrices $\delta_{ij} + Q_{ij}$ such that

$$\|Q\|_k = \sum_{i,j} (i+j)^k |Q_{ij}| < \infty, \quad \forall k \in \mathbb{N}_0,$$

the topology of $\mathcal{G}^{-\infty}$ being the one induced by the norms $\|\cdot\|_k$, $k \in \mathbb{N}_0$.

This isomorphism indicates that the topology of $G^{-\infty}(X; E)$ does not depend at all on the geometry of X and E . Since the direct limit

$$\text{GL}(\infty, \mathbb{C}) = \lim_{k \rightarrow \infty} \text{GL}(k, \mathbb{C})$$

is a weak deformation retract (see definition 7.16 below) of $\mathcal{G}^{-\infty}$, they share the same homotopy groups, which is to say

$$\pi_k(G^{-\infty}(X; E)) \cong \pi_k(\mathcal{G}^{-\infty}) \cong \begin{cases} \{0\} & k \text{ even,} \\ \mathbb{Z} & k \text{ odd.} \end{cases}$$

This periodicity in the homotopy groups is an instance of Bott periodicity as originally described by Bott in [4].

The author was partially supported by Le Fonds Québécois de la recherche sur la nature et les technologies and the Natural Sciences and Engineering Research Council of Canada.

In this paper, we will describe how Bott periodicity arises when one considers instead fibred cusp operators $\Psi_{\Phi}^*(X; E)$ on a compact manifold with boundary X acting on some complex vector bundle E . These operators were introduced by Mazzeo and Melrose in [11]. The definition involves a defining function for the boundary ∂X and a fibration $\Phi : \partial X \rightarrow Y$ of the boundary. Again, one considers the group

$$G_{\Phi}^{-\infty}(X; E) = \{\text{Id} + Q \mid Q \in \Psi_{\Phi}^{-\infty}(X; E), \text{Id} + Q \text{ is invertible}\}$$

of invertible smoothing perturbations of the identity. As before, it has a \mathcal{C}^{∞} -topology induced by the identification of smoothing operators with their Schwartz kernels. **Our main result, stated in theorem 7.19**, is to describe the homotopy groups of $G_{\Phi}^{-\infty}(X; E)$ in terms of the K -theory of T^*Y , namely, all the even homotopy groups are shown to be isomorphic to the kernel of the topological index map

$$\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z},$$

while all the odd homotopy groups are shown to be isomorphic to $\tilde{K}^{-1}(Y^{T^*Y})$, where Y^{T^*Y} is the Thom space associated to the vector bundle T^*Y . This periodicity of the homotopy groups is what we interpret as Bott periodicity for fibred cusp operators. Strictly speaking, this result is only true when the fibres of the fibration $\Phi : \partial X \rightarrow Y$ are of dimension at least one, but in the particular case where $\Phi : \partial X \rightarrow Y$ is a finite covering, which includes the case of scattering operators, the result is still true provided one allows some stabilization (see the discussion at the beginning of section 7).

This is in a certain sense a generalization of proposition 3.6 in [17], where it was shown that, in the particular case where $\Phi : \partial X \rightarrow \text{pt}$ is a trivial fibration (the case of cusp operators), all the homotopy groups of $G_{\Phi}^{-\infty}(X; E)$ are trivial. This weak contractibility was used in [17] to derive a relative index theorem for families of elliptic cusp pseudodifferential operators. However, this relative index theorem does not seem to generalize in a simple way to fibred cusp operators with a non-trivial fibration $\Phi : \partial X \rightarrow Y$, but we still hope that the knowledge of the homotopy groups of $G_{\Phi}^{-\infty}(X; E)$ will turn out to be useful in the understanding of the index of general Fredholm fibred cusp operators.

An important feature of fibred cusp operators is that smoothing operators are not necessarily compact. As a consequence, smoothing perturbations of the identity are not necessarily Fredholm, and when they are, they do not necessarily have a vanishing index. Our computation of the homotopy groups of $G_{\Phi}^{-\infty}(X; E)$ relies on a careful study of the space

$$\mathcal{F}_{\Phi}^{-\infty}(X; E) = \{\text{Id} + Q \mid Q \in \Psi_{\Phi}^{-\infty}(X; E) \text{ Id} + Q \text{ is Fredholm}\}$$

of Fredholm smoothing perturbations of the identity. **The second important result of this paper, stated in theorem 6.6**, is that the index of a Fredholm operator $(\text{Id} + Q) \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ can be described in terms of an associated K -class $\kappa(\text{Id} + Q) \in K_c^0(T^*Y)$, namely

$$\text{ind}(\text{Id} + Q) = \text{ind}_t(\kappa(\text{Id} + Q)),$$

where ind_t is the topological index introduced by Atiyah and Singer in [3]. When $\Phi : \partial X \rightarrow Y$ is not a finite covering, an important corollary is that the index map

$$\text{ind} : \mathcal{F}_{\Phi}^{-\infty}(X; E) \rightarrow \mathbb{Z}$$

is surjective. However, let us emphasize that the topology of $\mathcal{F}_{\Phi}^{-\infty}(X; E)$ is in general significantly different from the topology of the space $\mathcal{F}(X; E)$ of all Fredholm operators acting on $L^2(X; E)$. Indeed, $\mathcal{F}(X; E)$, which is a classifying space for even K -theory, has trivial odd homotopy groups and even homotopy groups isomorphic to \mathbb{Z} , while $\mathcal{F}_{\Phi}^{-\infty}(X; E)$, as described in proposition 7.6 below, has even homotopy groups isomorphic to $K_c^0(T^*Y)$ and odd homotopy groups isomorphic to $\tilde{K}^{-1}(Y^{T^*Y})$. Nevertheless, in the particular case of cusp operators ($Y = \text{pt}$), this shows that $\mathcal{F}_{\Phi}^{-\infty}(X; E) \subset \mathcal{F}(X; E)$ is quite big and is also a classifying space for even K -theory.

The paper is organized as follows. We first briefly review the definition and the main properties of fibred cusp operators in section 1. We then recall in section 2 the notion of a regularized trace on smoothing fibred cusp operators and its associated trace-defect formula. This is used in section 3 to get an analytic formula for the index of operators in $\mathcal{F}_{\Phi}^{-\infty}(X; E)$. Section 4 is about spectral sections, which play a crucial rôle in section 5, where a K -class is associated to any Fredholm operator in $\mathcal{F}_{\Phi}^{-\infty}(X; E)$. This K -class is in turn used in section 6 to get a topological index formula. This will allow us, in section 7, to present the main result of this paper, that is, the periodicity of the homotopy groups of $G_{\Phi}^{-\infty}(X; E)$. Finally, in section 8, we discuss the relation of our result with [17] and show that, once the weak contractibility is known, for instance in the case of cusp operators, it is not much harder, adapting a proof of Kuiper in [8], to deduce the actual contractibility, this last improvement being more esthetic than useful.

ACKNOWLEDGMENTS

The content of this paper is included in the Ph.D. thesis of the author. The author is very grateful to his Ph.D. advisor Richard B. Melrose, who was very generous of his time and ideas. The author would like to thank András Vasy for useful comments on the manuscript. The author would like also to thank Benoit Charbonneau, Kári Ragnarsson and Nataša Šešum for helpful discussions.

1. FIBRED CUSP OPERATORS

Since fibred cusp operators are the central object of study in this paper, let us recall briefly their definition and their main properties. A detailed description can be found in [11], where they were originally introduced. One can also look at section 2 of [10]. For a more general discussion in terms of Lie algebroids, we refer to [1] and [18].

Let X be a compact connected manifold with non-empty boundary ∂X . Assume that the boundary ∂X has the structure of a (locally trivial) fibration

$$(1.1) \quad \begin{array}{ccc} Z & \longrightarrow & \partial X \\ & & \downarrow \Phi \\ & & Y \end{array}$$

where Y and Z are closed manifolds, Y being the base of the fibration and Z being a typical fibre. We will not assume that the boundary ∂X is connected, but if Y is disconnected, we will assume that Z is the same over all connected components of Y . For the disconnected case, we will basically use the generalization of [17] rather than the one presented in [11] (see remark 1.3 below). Let $x \in \mathcal{C}^{\infty}(X)$ be a defining

function for the boundary ∂X , that is, x is non-negative, ∂X is the zero set of x and dx does not vanish anywhere on ∂X .

The defining function x gives rise to an explicit trivialization of the conormal bundle of ∂X . A fibration (1.1) together with an explicit trivialization of the conormal bundle of ∂X defines a fibred cusp structure. To a given fibred cusp structure, we will associate an algebra of fibred cusp operators denoted $\Psi_\Phi^*(X)$. Changing the trivialization of the conormal bundle of ∂X will not change $\Psi_\Phi^*(X)$ drastically, since we will get an isomorphic algebra, but not in a canonical way. When we talk about an algebra of fibred cusp operators, we therefore tacitly assume that a defining function has been chosen. On the other hand, changing the fibration (1.1) has important consequences on the algebra $\Psi_\Phi^*(X)$. In particular, there are two extreme cases: the case where $Z = \{\text{pt}\}$ and $\Phi = \text{Id}$, which leads to the algebra of **scattering operators**, and the case where $Y = \{\text{pt}\}$, which leads to the algebra of **cusp operators**.

Definition 1.1. A **fibred cusp vector field** $V \in \mathcal{C}^\infty(X, TX)$ is a vector field which, at the boundary ∂X , is tangent to the fibres of the fibration Φ , and such that $Vx \in x^2\mathcal{C}^\infty(X)$, where x is the defining function of the fibred cusp structure. We denote by $\mathcal{V}_\Phi(X)$ the Lie algebra of fibred cusp vector fields.

Let (x, y, z) be coordinates in a neighborhood of $p \in \partial X \subset X$, where x is the defining function for ∂X and y and z are respectively local coordinates on Y and Z (we assume that the fibration Φ is trivial on the neighborhood of p we consider). Then any fibred cusp vector field $V \in \mathcal{V}_\Phi(X)$ is locally of the form

$$(1.2) \quad V = ax^2 \frac{\partial}{\partial x} + \sum_{i=1}^l b_i x \frac{\partial}{\partial y^i} + \sum_{i=1}^k c_i \frac{\partial}{\partial z^i},$$

where $l = \dim Y$, $k = \dim Z$ and a, b_i, c_i are smooth functions.

Definition 1.2. The space of **fibred cusp differential operators** of order m , denoted $\text{Diff}_\Phi^m(X)$, is the space of operators on $\mathcal{C}^\infty(X)$ generated by $\mathcal{C}^\infty(X)$ and products of up to m elements of $\mathcal{V}_\Phi(X)$.

In the local coordinates (x, y, z) , a fibred cusp differential operator $P \in \text{Diff}_\Phi^m(X)$ may be written as

$$(1.3) \quad P = \sum_{|\alpha|+|\beta|+q \leq m} p_{\alpha,\beta,q} (x^2 \frac{\partial}{\partial x})^q (x \frac{\partial}{\partial y})^\alpha (\frac{\partial}{\partial z})^\beta$$

where the $p_{\alpha,\beta,q}$ are smooth functions.

Intuitively, fibred cusp pseudodifferential operators are a generalization of definition 1.2 by allowing P to be not only a polynomial in $x^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, but also a more general “function” in $x^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. The rigorous definition is however better understood by describing the Schwartz kernels of fibred cusp pseudodifferential operators.

Consider the cartesian product X^2 of the manifold X with itself. This is a manifold with corner, the corner being $\partial X \times \partial X \subset X^2$. Schwartz kernels of fibred cusp pseudodifferential operators will be distributions on X^2 . But all the subtleties of their behavior turn out to happen in the corner of X^2 . To have a better picture of what is going on, one therefore blows up the corner in the sense of Melrose, [12], [13]. In fact, we need to blow up twice.

We first blow up the corner to get the space $X_b^2 = [X^2; \partial X \times \partial X]$. In [13], this space is used to describe the Schwartz kernels of b -pseudodifferential operators, where “ b ” stands for boundary. For fibred cusp operators, we need to perform a second blow-up which depends on the fibration Φ we consider.

Beside the two old boundaries coming from X^2 , X_b^2 has a new boundary called the **front face**. If $\beta_b : X_b^2 \rightarrow X^2$ is the blow-down map, then this front face, denoted ff_b , is given by

$$(1.4) \quad \text{ff}_b = \beta_b^{-1}(\partial X \times \partial X) \cong (\partial X \times \partial X) \times [-1, 1]_s,$$

where $s = \frac{x-x'}{x+x'}$ and x, x' are the same boundary defining functions on the left and right factors of X^2 . The function s is well-defined on X_b^2 . What we blow up next is the submanifold $B_\Phi \subset \text{ff}_b \cong (\partial X \times \partial X) \times [-1, 1]_s$ given by

$$(1.5) \quad B_\Phi = \{(h, h', 0) \in \text{ff}_b \mid \Phi(h) = \Phi(h')\}.$$

Remark 1.3. When ∂X is disconnected, we use the same definition for B_Φ and ff_b . This is different from [11], where they consider smaller B_Φ and ff_b in the disconnected case. Using the same formal definition as in the connected case makes the generalization to the disconnected case straightforward.

So we consider the space $X_\Phi^2 = [X_b^2; B_\Phi]$. If $\beta_\Phi : X_\Phi^2 \rightarrow X_b^2$ is the blow-down map, then the new boundary appearing on X_Φ^2 is given by

$$(1.6) \quad \text{ff}_\Phi = \beta_\Phi^{-1}(B_\Phi).$$

If (y, z) and (y', z') are local coordinates on the left and right factors of $(\partial X)^2$, then

$$(1.7) \quad S = \frac{x-x'}{x^2}, Y = \frac{y-y'}{x}, z-z', x, y, z$$

are local coordinates on X_Φ^2 , and in these coordinates, ff_Φ occurs where $x = 0$. Under the blow-down maps β_b and β_Φ , the diagonal $\Delta \subset \partial X \times \partial X$ lifts to $\Delta_\Phi \subset X_\Phi^2$. In the coordinates (1.7), it occurs where $S = Y = z - z' = 0$.

Definition 1.4. The **fibred cusp density bundle** ${}^\Phi\Omega$ is the space of densities on X which are locally of the form $\frac{\alpha}{x^{l+2}} dx dy dz$ near the boundary ∂X , where α is a smooth function and $l = \dim Y$. Let ${}^\Phi\Omega_R$ be the lift of ${}^\Phi\Omega$ to X^2 from the right factor. This gives the **right fibred cusp density bundle** ${}^\Phi\Omega'_R = \beta_\Phi^* \beta_b^* ({}^\Phi\Omega_R)$ on X_Φ^2 .

Definition 1.5. For any $m \in \mathbb{R}$, the space of Φ -pseudodifferential operators of order m is given by

$$\Psi_\Phi^m(X) = \{K \in I^m(X_\Phi^2, \Delta_\Phi; {}^\Phi\Omega'_R) \mid K \equiv 0 \text{ at } \partial X_\Phi^2 \setminus \text{ff}_\Phi\}$$

where $I^m(X_\Phi^2, \Delta_\Phi; {}^\Phi\Omega'_R)$ denotes the space of conormal distributions (to Δ_Φ) of order m (we refer to [11] and [7] for more details) and \equiv denotes equality in Taylor series.

In local coordinates near the boundary, the action (near the diagonal) of fibred cusp pseudodifferential operators can be described as follows.

Proposition 1.6. *If $\chi \in \mathcal{C}^\infty(X)$ has support in a coordinate patch $(\mathcal{U}, (x, y, z))$ with $\mathcal{U} \cap \partial X \neq \emptyset$, where x is the defining function of ∂X and y and z are coordinates on Y and Z , then the action of $P \in \Psi_\Phi^m(X)$ on $u \in \mathcal{C}_c^\infty(\mathcal{U})$ can be written*

$$(\chi Pu)(x, y, z) = \int P_\chi(x, y, z, S, Y, z - z') \tilde{u}(x(1 - xS), y - xY, z') dS dY dz'$$

where $S = \frac{x-x'}{x^2}$, $Y = \frac{y-y'}{x}$, \tilde{u} is the coordinate representation of u and P_χ is the restriction to $\mathcal{U} \times \mathbb{R}^n$ of a distribution on $\mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^k$ which has compact support in the first and third variables, is conormal to $\{S = 0, Y = 0\} \times \{z = z'\}$ and is rapidly decreasing with all derivatives as $|(S, Y)| \rightarrow \infty$.

If E and F are complex vector bundles on X , one can easily extend this definition to Φ -pseudodifferential operators $\Psi_\Phi^m(X; E, F)$ of order m which act from $\mathcal{C}^\infty(X; E)$ to $\mathcal{C}^\infty(X; F)$.

Proposition 1.7. (Composition) *For $m, n \in \mathbb{R}$ and E, F, G complex vector bundles on X , we have that*

$$\Psi_\Phi^m(X; F, G) \circ \Psi_\Phi^n(X; E, F) \subset \Psi_\Phi^{m+n}(X; E, G).$$

As in the case of usual pseudodifferential operators, there is a notion of ellipticity.

Definition 1.8. *The **fibred cusp tangent bundle** ${}^\Phi TX$ is the smooth vector bundle on X such that $\mathcal{V}_\Phi(X) = \mathcal{C}^\infty(X; {}^\Phi TX)$.*

Notice that ${}^\Phi TX$ is isomorphic to TX , although not in a natural way. Let ${}^\Phi S^*X = ({}^\Phi TX \setminus 0)/\mathbb{R}^+$ denote the fibred cusp sphere bundle. Let \mathcal{R}^m be the trivial complex line bundle on ${}^\Phi S^*X$ with sections given by functions over ${}^\Phi TX \setminus 0$ which are positively homogeneous of degree m .

Proposition 1.9. *For all $m \in \mathbb{Z}$, there exists a symbol map*

$$\sigma_m : \Psi_\Phi^m(X; E, F) \rightarrow \mathcal{C}^\infty({}^\Phi S^*X; \text{hom}(E, F) \otimes \mathcal{R}^m)$$

such that we have the following short exact sequence

$$\Psi_\Phi^{m-1}(X; E, F) \longrightarrow \Psi_\Phi^m(X; E, F) \xrightarrow{\sigma_m} \mathcal{C}^\infty({}^\Phi S^*X; \text{hom}(E, F) \otimes \mathcal{R}^m).$$

Definition 1.10. *If $P \in \Psi_\Phi^m(X; E, F)$, we say that $\sigma_m(P)$ is the **symbol** of P . Moreover, if the symbol $\sigma_m(P)$ is invertible, we say that P is **elliptic**.*

As opposed to the case of usual pseudodifferential operators, ellipticity is not a sufficient condition to ensure that an operator is Fredholm. Some control near the boundary is also needed. Let ${}^\Phi NY \cong TY \times \mathbb{R}$ be the null bundle on Y of the restriction ${}^\Phi T_{\partial X} X \rightarrow TX$.

Proposition 1.11. *For any $m \in \mathbb{R}$, there is a short exact sequence*

$$x\Psi_\Phi^m(X; E, F) \longrightarrow \Psi_\Phi^m(X; E, F) \xrightarrow{N_\Phi} \Psi_{\Phi_s}^m(\partial X; E, F)$$

where N_Φ is the **normal operator** and $\Psi_{\Phi_s}^m(\partial X; E, F)$ is the space of ${}^\Phi NY$ -suspended fibre pseudodifferential operators on ∂X of order m (see definition 2 in [11]). The normal operator is multiplicative, that is $N_\Phi(A) \circ N_\Phi(B) = N_\Phi(A \circ B)$ whenever A and B compose. In terms of the Schwartz kernels, the normal operator N_Φ is just the **restriction of the Schwartz kernel to the front face** ff_Φ .

Definition 1.12. An operator $P \in \Psi_{\Phi}^m(X; E, F)$ is said to be **fully elliptic** if it is elliptic and if $N_{\Phi}(P)$ is invertible.

Definition 1.13. Let $L^2(X; E)$ be the space of square integrable sections of E with respect to some metric on E and some smooth positive density on X . For $l \in \mathbb{R}$ and $m \geq 0$, we define the associated weighted Sobolev spaces by

$$(1.8) \quad x^l H_{\Phi}^m(X; E) = \{u \in x^l L^2(X; E) \mid Pu \in x^l L^2(X; E) \forall P \in \Psi_{\Phi}^m(X; E)\},$$

and

$$(1.9) \quad x^l H_{\Phi}^{-m}(X; E) = \left\{ u \in C^{-\infty}(X; E) \mid u = \sum_{i=1}^N P_i u_i, u_i \in x^l L^2(X; E), P_i \in \Psi_{\Phi}^m(X; E) \right\}.$$

Proposition 1.14. An operator $P \in \Psi_{\Phi}^m(X; E, F)$ is Fredholm as a map

$$P : x^l H_{\Phi}^{m'}(X; E) \rightarrow x^l H_{\Phi}^{m'-m}(X; F)$$

for any $l, m \in \mathbb{R}$ if and only if it is fully elliptic.

Proposition 1.15. (elliptic regularity) The null space of a fully elliptic operator $P \in \Psi_{\Phi}^m(X; E, F)$ is contained in $\dot{C}^{\infty}(X; E)$, the algebra of smooth sections of E vanishing in Taylor series at the boundary ∂X .

It would be unfair to end this section without discussing in more details the space of suspended operators $\Psi_{\Phi_s}^*(\partial X; E, F)$. Since this paper is mostly concerned with smoothing operators, we will restrict our attention to $\Psi_{\Phi_s}^{-\infty}(\partial X; E, F)$. By taking the **Fourier transform** in the fibres of ${}^{\Phi}NY$, it is possible to describe $P \in \Psi_{\Phi_s}^{-\infty}(\partial X; E, F)$ as a smooth family of smoothing operators parametrized by ${}^{\Phi}N^*Y$, the dual of ${}^{\Phi}NY$. More precisely, consider the bundle

$$(1.10) \quad \begin{array}{ccc} \Psi^{-\infty}(Z; E, F) & \longrightarrow & \mathcal{P}^{-\infty} \\ & & \downarrow \\ & & Y \end{array}$$

whose fibre at $y \in Y$ is given by $\Psi^{-\infty}(\Phi^{-1}(y); E|_{\Phi^{-1}(y)}, F|_{\Phi^{-1}(y)})$.

Remark 1.16. When $\dim Z = 0$ and $\Phi : \partial X \rightarrow Y$ is a finite covering, the bundle $\mathcal{P}^{-\infty}$ is the vector bundle $\text{hom}(\mathcal{E}, \mathcal{F})$, where $\mathcal{E} \rightarrow Y$ and $\mathcal{F} \rightarrow Y$ are the complex vector bundles over Y with fibres at $y \in Y$ given by

$$\mathcal{E}_y = \bigoplus_{z \in \Phi^{-1}(y)} E_z, \quad \mathcal{F}_y = \bigoplus_{z \in \Phi^{-1}(y)} F_z.$$

We can pull back $\mathcal{P}^{-\infty}$ on ${}^{\Phi}N^*Y$ to get a bundle $\pi^* \mathcal{P}^{-\infty} \rightarrow {}^{\Phi}N^*Y$, where $\pi : {}^{\Phi}N^*Y \rightarrow Y$ is the projection associated to the vector bundle ${}^{\Phi}N^*Y$.

Proposition 1.17. There is a one-to-one correspondence between $\Psi_{\Phi_s}^{-\infty}(\partial X; E, F)$ and smooth sections of $\pi^* \mathcal{P}^{-\infty} \rightarrow {}^{\Phi}N^*Y$ which are **rapidly decreasing** with all derivatives as one approaches infinity in ${}^{\Phi}N^*Y \cong T^*Y \times \mathbb{R}$.

Remark 1.18. For $P \in \Psi_{\Phi_s}^m(\partial X; E, F)$, one has a similar correspondence, but one must replace $\pi^*\mathcal{P}^{-\infty}$ by the appropriate bundle $\pi^*\mathcal{P}^m$ of pseudodifferential operators of order m , while the sections at infinity should grow no faster than a homogeneous function of order m . The derivatives of the sections must also satisfy some growing conditions at infinity.

This is the point of view we will adopt for the rest of this paper, that is, we will consider the Fourier transform of $N_{\Phi}(P)$, which is called the **indicial family** of P , instead of $N_{\Phi}(P)$. One advantage of doing so is that it is relatively easy to understand the indicial family in terms of local coordinates as in (1.7). If $P \in \Psi_{\Phi}^{-\infty}(X; E, F)$ acts locally as in proposition 1.6 and if τ, η are the dual variables to x, y , then the indicial family is given by

$$(1.11) \quad \widehat{N_{\Phi}(P)}(y, \eta, \tau) = \int e^{i\eta \cdot Y} e^{i\tau S} P(0, y, z, S, Y, z - z') dS dY.$$

2. THE TRACE-DEFECT FORMULA

For usual pseudodifferential operators on a closed manifold, it is a well-known fact that an operator will be of trace class provided its order is sufficiently low. For fibred cusp pseudodifferential operators, this is no longer true. Even smoothing operators are not necessarily of trace class. This is because there is a potential singularity at the boundary.

Let us concentrate on the case of smoothing operators. So we consider the space $\Psi_{\Phi}^{-\infty}(X; E)$, where E is some complex vector bundle on X . To avoid possible ambiguities, we assume that in a collar neighborhood $\partial X \times [0, 1)_x \subset X$ of ∂X parametrized by the defining function x , the vector bundle E is of the form $E = \pi^*E_{\partial X}$, where $E_{\partial X}$ is a vector bundle on ∂X and $\pi : \partial X \times [0, 1)_x \rightarrow \partial X$ is the projection on the first factor. In other words, we choose an **explicit trivialization** of E near the boundary which agrees with the choice of the defining function x . This way, we can make sense of Taylor series at the boundary in powers of x (see (2.3) below).

If $A \in \Psi_{\Phi}^{-\infty}(X; E)$ is of trace class, then its trace is given by integrating its Schwartz kernel along the diagonal

$$(2.1) \quad \text{Tr}(A) = \int_{\Delta_{\Phi}} \text{tr}_E K_A = \int_{\Delta_{\Phi}} \text{tr}_E (K'_A) x^{-l-2} \nu, \quad K'_A \in \mathcal{C}^{\infty}(X_{\Phi}^2; \text{Hom}(E, E)),$$

where $l = \dim Y$, $(x')^{-l-2} \nu$ is a smooth section of the right fibred cusp density bundle ${}^{\Phi}\Omega'_R$, and $\text{Hom}(E, E)$ is the pull back under the blow-down map $\beta_{\Phi} \circ \beta_b$ of the homomorphism bundle on X^2 with fibre at $(p, p') \in X^2$ given by $\text{hom}(E_{p'}, E_p)$. When restricted to the diagonal $\Delta \subset X^2$, $\text{Hom}(E, E)$ is the same as $\text{hom}(E, E)$.

The integral (2.1) does not converge for a general $A \in \Psi_{\Phi}^{-\infty}(X; E)$ since ν is a smooth positive density when restricted to the diagonal Δ_{Φ} . To extend the trace to all smoothing operators, one has to subtract the source of divergence. To this end, consider the function

$$(2.2) \quad z \mapsto \text{Tr}(x^z A),$$

which is a priori only defined for $\text{Re}\{z\} > l + 1$.

Lemma 2.1. *The function $\text{Tr}(x^z A)$ is holomorphic for $\text{Re}\{z\} > l+1$ and it admits a meromorphic extension to the whole complex plane with at most simple poles at $l+1 - \mathbb{N}_0$.*

Proof. The holomorphy is clear. Recall from proposition 1.11 that in terms of the Schwartz kernels, the normal operator is the restriction to the front face ff_Φ . Let us consider instead the **full normal operator** N'_Φ which gives us the full Taylor series of the Schwartz kernel at the front face ff_Φ , using x as a defining function,

$$(2.3) \quad N'_\Phi(A) = \sum_{k=0}^{\infty} x^k A_k, \quad A_k \in \Psi_{\Phi_s}^{-\infty}(\partial X; E).$$

This Taylor series does not necessarily converge, so $N'_\Phi(A)$ is really a formal power series in x which contains all the information about the derivatives of A at the front face ff_Φ . The Taylor series $N'_\Phi(A)$ contains all the terms causing a divergence in the definition of $\text{Tr}(x^z A)$. A simple computation shows that the function $\text{Tr}(x^z x^k A_k \chi(x))$, which is a priori only defined for $\text{Re}\{z\} > l+1-k$, has a meromorphic extension to the whole complex plane with a single simple pole at $z = l+1-k$. Here, $\chi \in \mathcal{C}_c^\infty([0, +\infty))$ is some cut-off function with $\chi \equiv 1$ in a neighborhood of $x = 0$. Thus, we see that $\text{Tr}(x^z A)$ admits a meromorphic extension with at most simple poles at $l+1 - \mathbb{N}_0$. \square

In particular, there is in general a pole at $z = 0$ coming from the term A_{l+1} in (2.3).

Definition 2.2. *For $A \in \Psi_\Phi^{-\infty}(X; E)$, the **residue trace** $\text{Tr}_R(A)$ of A is the residue at $z = 0$ of the meromorphic function $z \mapsto \text{Tr}(x^z A)$.*

Using the local representation of proposition 1.6, the residue trace can be expressed as

$$(2.4) \quad \text{Tr}_R(A) = \int \text{tr}_E(A_{l+1}(y, z, S=0, Y=0, z-z'=0)) dy dz.$$

Taking the Fourier transform in the fibres of $N_\Phi Y \cong TY \times \mathbb{R}$ as in (1.11), the residue trace can be rewritten as (cf. equations 4.1 and 4.2 in [14])

$$(2.5) \quad \text{Tr}_R(A) = \frac{1}{(2\pi)^{l+1}} \int_{N_\Phi^* Y} \text{Tr}_{\Phi^{-1}(y)}(\hat{A}_{l+1}(y, \tau, \eta)) dy d\eta d\tau$$

where $\text{Tr}_{\Phi^{-1}(y)}$ is the trace on $\Psi^{-\infty}(\Phi^{-1}(y); E|_{\Phi^{-1}(y)})$. Notice that this integral does not depend on the choice of coordinates on Y , since $dy d\eta$ is the volume form coming from the canonical symplectic form on T^*Y . Of course, $d\tau$ depends on the choice of the defining function x , but we assume it has been fixed.

One can then extend the trace to all smoothing operators in $\Psi_\Phi^{-\infty}(X; E)$ by subtracting the pole at $z = 0$.

Definition 2.3. *The **regularized trace** $\overline{\text{Tr}}(A)$ of $A \in \Psi_\Phi^{-\infty}(X; E)$ is given by*

$$\overline{\text{Tr}}(A) = \lim_{z \rightarrow 0} \left(\text{Tr}(x^z A) - \frac{\text{Tr}_R(A)}{z} \right).$$

Remark 2.4. *For $A \in x^{l+2} \Psi_\Phi^{-\infty}(X; E)$, A is of trace class and $\overline{\text{Tr}}(A) = \text{Tr}(A)$.*

The regularized trace is defined for all $A \in \Psi_{\Phi}^{-\infty}(X; E)$ and it is an extension of the usual trace, but it is not properly speaking a trace, since it does not vanish on commutators. The trace-defect formula measures how far the regularized trace is from being a trace.

Proposition 2.5. (Trace-defect formula) *For $A, B \in \Psi_{\Phi}^{-\infty}(X; E)$, the trace-defect is given by*

$$\overline{\text{Tr}}([A, B]) = \text{Tr}_{\text{R}}((D_{\log x} A)B)$$

where $D_{\log x} A = \frac{\partial}{\partial z} A_z \Big|_{z=0}$ and $A_z = x^{-z} [x^z, A]$.

Proof. For $\text{Re}\{z\} \gg 0$, we have

$$\begin{aligned} \text{Tr}(x^z [A, B]) &= \text{Tr}(x^z AB - x^z BA) = \text{Tr}(x^z AB) - \text{Tr}((x^z B)A) \\ (2.6) \quad &= \text{Tr}(x^z AB) - \text{Tr}(Ax^z B) - \text{Tr}([x^z B, A]). \end{aligned}$$

But $\text{Tr}([x^z B, A]) = 0$ for $\text{Re}\{z\} \gg 0$, so

$$\begin{aligned} \text{Tr}(x^z [A, B]) &= \text{Tr}(x^z AB) - \text{Tr}(Ax^z B) \\ (2.7) \quad &= \text{Tr}([x^z, A]B) = \text{Tr}(x^z A_z B). \end{aligned}$$

Here, $A_z = x^{-z} [x^z, A]$ is entire in z and vanishes at $z = 0$, so

$$(2.8) \quad \text{Tr}(x^z [A, B]) = z \text{Tr}(x^z (D_{\log x} A)B) + \mathcal{O}(z), \quad \text{where} \quad D_{\log x} A = \frac{\partial}{\partial z} A_z \Big|_{z=0}.$$

Thus, we see that $\text{Tr}(x^z [A, B])$ has no pole at $z = 0$ and its value at $z = 0$ is given by

$$\overline{\text{Tr}}([A, B]) = \text{Tr}_{\text{R}}((D_{\log x} A)B).$$

□

In the case of cusp operators ($Y = \{\text{pt}\}$), it is possible to use (2.5) to get an explicit formula depending only on the indicial families of A and B (cf. [17]). In the more general case, higher order terms of the Taylor series (2.3) are involved. Combined with the geometry of Y , this makes an explicit computation harder or maybe even impossible to get.

3. THE INDEX IN TERMS OF THE TRACE-DEFECT FORMULA

Let $\mathcal{F}_{\Phi}^{-\infty}(X; E)$ denote the space of Fredholm operators of the form $\text{Id} + A$ with $A \in \Psi_{\Phi}^{-\infty}(X; E)$. From proposition 1.14, the space $\mathcal{F}_{\Phi}^{-\infty}(X; E)$ is given by

$$(3.1) \quad \mathcal{F}_{\Phi}^{-\infty}(X; E) = \{\text{Id} + A \mid A \in \Psi_{\Phi}^{-\infty}(X; E), \ N_{\Phi}(\text{Id} + A) \text{ is invertible}\}.$$

The goal of the next few sections will be to obtain a topological description of the index of operators in $\mathcal{F}_{\Phi}^{-\infty}(X; E)$, that is, a description in terms of K -theory. As a first step, let us use the trace-defect formula to get some insight about the index.

Notation 3.1. *From now on we will follow the notation of (2.3) for the indicial family. So $\widehat{A}_0 = \widehat{N_{\Phi}(A)}$ will denote the indicial family. In proposition 1.17, it is \widehat{A}_0 which can be seen as a section of the bundle $\pi^* \mathcal{P}^{-\infty}$.*

Given $(\text{Id} + A) \in \mathcal{F}_\Phi^{-\infty}(X; E)$, let $(\text{Id} + B) \in \mathcal{F}_\Phi^{-\infty}(X; E)$ be a parametrix, that is

$$(3.2) \quad (\text{Id} + A)(\text{Id} + B) = \text{Id} + Q_1, \quad (\text{Id} + B)(\text{Id} + A) = \text{Id} + Q_2,$$

with $Q_1, Q_2 \in x^\infty \Psi_\Phi^{-\infty}(X; E)$, where

$$(3.3) \quad x^\infty \Psi_\Phi^{-\infty}(X; E) = \{Q \in \Psi_\Phi^{-\infty}(X; E) \mid N'_\Phi(Q) = \sum_{k=0}^{\infty} x^k Q_k = 0\}.$$

In particular, Q_1 and Q_2 are compact operators of trace class, so $[\text{Id} + A, \text{Id} + B] = Q_1 - Q_2$ is also of trace class. Using Calderón's formula for the index, we have that

$$(3.4) \quad \begin{aligned} \text{ind}(\text{Id} + A) &= \text{Tr}([\text{Id} + A, \text{Id} + B]) \\ &= \overline{\text{Tr}}([\text{Id} + A, \text{Id} + B]), \quad \text{by remark 2.4} \\ &= \overline{\text{Tr}}([A, B]) \\ &= \text{Tr}_R((D_{\log x} A)B), \quad \text{by proposition 2.5.} \end{aligned}$$

Using (2.5), we get the following formula for the index.

Proposition 3.2. *The index of $(\text{Id} + A) \in \mathcal{F}_\Phi^{-\infty}(X; E)$ is given by*

$$\text{ind}(\text{Id} + A) = \frac{1}{(2\pi)^{l+1}} \int_{\Phi \circ N^* Y} \text{Tr}_{\Phi^{-1}(y)}((\widehat{D_{\log x} A} B)_{l+1}(y, \tau, \eta)) dy d\eta d\tau$$

where $(\text{Id} + B) \in \mathcal{F}_\Phi^{-\infty}(X; E)$ is a parametrix for A .

In the case of cusp operators, one can rewrite this formula in terms of the indicial family of A only (see [17]). In fact, one can show without using the trace-defect formula that the index only depends on the indicial family.

Proposition 3.3. *The index of $(\text{Id} + A) \in \mathcal{F}_\Phi^{-\infty}(X; E)$ only depends on the homotopy class of the indicial family $\text{Id} + \hat{A}_0$ seen as an **invertible** section of the bundle $\pi^* \mathcal{P}^{-\infty}$ of proposition 1.17 which **converges rapidly** to Id with all derivatives as one approaches infinity.*

Proof. Suppose that $t \mapsto \text{Id} + \hat{A}_0(t)$, $t \in [0, 1]$ is a (smooth) homotopy of invertible indicial families with $\hat{A}_0(t) \in \Psi_{\Phi s}^{-\infty}(\partial X; E)$ for all $t \in [0, 1]$. Suppose also that $\text{Id} + \hat{A}_0(0)$ and $\text{Id} + \hat{A}_0(1)$ are the indicial families of $(\text{Id} + A(0))$, $(\text{Id} + A(1)) \in \mathcal{F}_\Phi^{-\infty}(X; E)$. Thinking in terms of Schwartz kernels and using Seeley extension for manifolds with corners (see the proof of proposition 1.4.1 in [12]), we can construct a homotopy $t \mapsto A(t)$, $t \in [0, 1]$, joining $A(0)$ and $A(1)$, such that $A(t) \in \Psi_\Phi^{-\infty}(X; E)$ and

$$\widehat{N_\Phi(A(t))} = \hat{A}_0(t) \quad \forall t \in [0, 1].$$

Since $t \mapsto \text{Id} + \hat{A}_0(t)$ is a homotopy of **invertible** indicial families, we know by proposition 1.14 that $t \mapsto (\text{Id} + A(t))$ is a homotopy of Fredholm operators. By the homotopy invariance of the index, we conclude that

$$\text{ind}(\text{Id} + A(0)) = \text{ind}(\text{Id} + A(1)),$$

which shows that the index of an operator $(\text{Id} + A) \in \mathcal{F}_\Phi^{-\infty}(X; E)$ only depends on the homotopy class of its indicial family. \square

One therefore expects that it is possible, at least in theory, to rewrite the formula of proposition 3.2 so that it only involves the indicial family. A first step in this direction is to notice that, by proposition 3.3, we can assume that the operator $\text{Id} + A \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ is such that

$$\widehat{N'_{\Phi}(A)} = \widehat{A}_0, \text{ with } A_k = 0 \ \forall k \in \mathbb{N}.$$

However, once this assumption is made on A , it is not possible to assume the same for the parametrix B . To be more precise, the equation (3.2) completely determined $\widehat{N'_{\Phi}(B)}$ in terms of $\widehat{N'_{\Phi}(A)}$. This will become clear through the concept of $*$ -product.

Definition 3.4. Let $\Psi_{\Phi_s}^{-\infty}(\partial X; E)[[x]]$ denote the space of formal power series in x with coefficients taking values in $\Psi_{\Phi_s}^{-\infty}(\partial X; E)$.

Lemma 3.5. The full normal operator $N'_{\Phi} : \Psi_{\Phi}^{-\infty}(X; E) \rightarrow \Psi_{\Phi_s}^{-\infty}(\partial X; E)[[x]]$ is surjective.

Proof. This is an easy consequence of Seeley extension for manifolds with corners. \square

Definition 3.6. For $a, b \in \Psi_{\Phi_s}^{-\infty}(\partial X; E)[[x]]$, the $*$ -product $a * b \in \Psi_{\Phi_s}^{-\infty}(\partial X; E)[[x]]$ is defined to be

$$a * b = N'_{\Phi}(\widehat{A \circ B})$$

where $A, B \in \Psi_{\Phi}^{-\infty}(X; E)$ are chosen such that $\widehat{N'_{\Phi}(A)} = a$, $\widehat{N'_{\Phi}(B)} = b$. It does not depend on the choice of A and B , since $x^{\infty}\Psi_{\Phi}^{-\infty}(X; E)$ is an ideal of $\Psi_{\Phi}^{-\infty}(X; E)$.

Proposition 3.7. There exist differential operators $P_{k,j,l}$ and $Q_{k,j,l}$ acting on sections of the bundle $\mathcal{P}^{-\infty}$ defined in (1.10) so that, for any $a, b \in \Psi_{\Phi_s}^{-\infty}(\partial X; E)[[x]]$, the $*$ -product of a and b is given by

$$a * b = \left(\sum_{j=0}^{\infty} x^j a_j \right) * \left(\sum_{j=0}^{\infty} x^j b_j \right) = \sum_{k=1}^{\infty} \sum_{j+l \leq k} x^k (P_{k,j,l} a_j) \circ (Q_{k,j,l} b_l).$$

Moreover, if $\Phi : \partial X \rightarrow Y$ is a trivial fibration, then, once an explicit trivialization has been chosen, the differential operators $P_{k,j,l}$ and $Q_{k,j,l}$ are independent of the geometry of the typical fibre Z .

For the proof of proposition 3.7, we refer to proposition 3.11 in [9]. In terms of the $*$ -product, equation (3.2) translates into

$$(3.5) \quad \widehat{N'_{\Phi}(A)} + \widehat{N'_{\Phi}(B)} = -\widehat{N'_{\Phi}(A)} * \widehat{N'_{\Phi}(B)} = -\widehat{N'_{\Phi}(B)} * \widehat{N'_{\Phi}(A)}.$$

Looking at this equation order by order in x completely determines $\widehat{N'_{\Phi}(B)}$ in terms of $\widehat{N'_{\Phi}(A)}$. In general, for $k \in \mathbb{N}$, one can expect that $\widehat{B}_k \neq 0$, so $\widehat{N'_{\Phi}(B)}$ will involve a term of order x^k . This is also the case for $D_{\log x} A$.

Lemma 3.8. If $\widehat{N'_{\Phi}(A)} = \widehat{A}_0$, then

$$N'_{\Phi}(\widehat{D_{\log x} A}) = \sum_{k=1}^{\infty} \frac{x^k D_{\tau}^k \widehat{A}_0}{k}.$$

Proof. The Schwartz kernel of $A_z = x^{-z}[x^z, A]$ is $A(1 - (\frac{x'}{x})^z)$, so the the Schwartz kernel of $D_{\log x}A$ is given by

$$\frac{\partial}{\partial z} \left(A(1 - (\frac{x'}{x})^z) \right) \Big|_{z=0} = -A \log \left(\frac{x'}{x} \right).$$

But $\frac{x'}{x} = 1 - Sx$, where $S = \frac{x-x'}{x^2}$, so

$$D_{\log x}A = -A \log(1 - Sx) = A \sum_{k=1}^{\infty} \frac{(Sx)^k}{k}.$$

From (1.11), $N_{\Phi}(\widehat{S^k A}) = D_{\tau}^k \widehat{A}_0$, where $D_{\tau} = -i \frac{\partial}{\partial \tau}$, so we conclude that

$$N'_{\Phi}(\widehat{D_{\log x} A}) = \sum_{k=1}^{\infty} \frac{x^k D_{\tau}^k \widehat{A}_0}{k}.$$

□

Thus, knowing the differential operators $P_{k,j,l}$ and $Q_{k,j,l}$, we see that it is possible to rewrite the formula of proposition 3.2 into an expression involving only the indicial family of A . The problem of course is to actually find out what are those differential operators. This is relatively easy in the cusp case, but the case of an arbitrary manifold Y seems hopeless. We will not try to develop further the formula of proposition 3.2. As it is written now and from proposition 3.7, one can extract some important information about the index. For instance, notice that the formula does not involve the geometry of the interior of X , so if X' is another manifold with the same boundary as X and if $(\text{Id} + A') \in \mathcal{F}_{\Phi}^{-\infty}(X'; E)$ has the same indicial family as the one for $\text{Id} + A$, then $\text{Id} + A$ and $\text{Id} + A'$ have the same index. In section 6, the formula of proposition 3.2 will play a crucial rôle in reducing the computation of the index to the case of a scattering operator. For the moment however, we will try to have a better understanding of the topological information contained in the indicial family. When the fibration Φ is trivial, it is almost straightforward to define a K -class out of the homotopy class of the indicial family. But to include the more general case of a non-trivial fibration, we first need to discuss the concept of spectral section.

4. SPECTRAL SECTIONS

Spectral sections were originally introduced in [16] to describe the boundary conditions for families of Dirac operators on manifolds with boundary. We intend to use spectral sections for a very different purpose, and instead of dealing with families of Dirac operators, we will consider families of Laplacians.

Let $\phi : M \rightarrow B$ be a smooth (locally trivial) fibration of compact manifolds, where the fibres are closed manifolds and the base B is a compact manifold with possibly a non-empty boundary ∂B . Let $T(M/B) \subset TM$ denote the null space of the differential

$$\phi_* : TM \rightarrow TB$$

of ϕ . Clearly, $T(M/B)$ is a subbundle of TM which on each fibre $M_b = \phi^{-1}(b)$ restricts to be canonically isomorphic to TM_b . Let $g_{M/B}$ be a family of metrics on M , that is, $g_{M/B}$ is a metric on $T(M/B)$ which gives rise to a Riemannian metric

on each fibre. Let E be a complex vector bundle with an Euclidean metric g^E and a connection ∇^E .

This allows us to define a smooth family of Laplacians $\Delta_{M/B}$, which on a fibre $M_b = \phi^{-1}(b)$ acts as

$$(4.1) \quad \Delta_b s = -\text{Tr}(\nabla^{T^*M_b \otimes E} \nabla^E s), \quad s \in \mathcal{C}^\infty(M_b; E_b), \quad E_b = E|_{M_b},$$

where $\nabla^{T^*M_b \otimes E}$ is induced from ∇^E and the Levi-Civita connection of $g_b = g_{M/B}|_{M_b}$, and $\text{Tr}(S) \in \mathcal{C}^\infty(M_b; E_b)$ is the contraction of an element

$$S \in \mathcal{C}^\infty(M_b; T^*M_b \otimes T^*M_b \otimes E_b)$$

with the metric $g_b \in \mathcal{C}^\infty(M_b, TM_b \otimes TM_b)$.

Let $L^2(M/B; E) \rightarrow B$ be the Hilbert bundle over B with fibre at $b \in B$ given by $L^2(M_b; E_b)$. The scalar product on $L^2(M_b; E_b)$ is defined in the usual way by

$$(4.2) \quad \langle s_1, s_2 \rangle = \int g^E(s_1, \bar{s}_2) dg_b, \quad s_1, s_2 \in \mathcal{C}^\infty(M_b; E_b),$$

where dg_b is the volume form associated to the metric g_b .

As is well-known, the Laplacian Δ_b is self-adjoint and has a non-negative discrete spectrum. Moreover, eigensections with different eigenvalues are orthogonal in $L^2(M_b; E_b)$.

Definition 4.1. A *spectral section* $(\Pi_{M/B}, R_1, R_2)$ for the family of Laplacians $\Delta_{M/B}$ is a smooth family $\Pi_{M/B} : L^2(M/B; E) \rightarrow L^2(M/B; E)$ of projections

$$\Pi_b : L^2(M_b; E_b) \rightarrow L^2(M_b; E_b), \quad b \in B,$$

with range a *trivial* vector bundle over B , together with real numbers R_1 and R_2 , $R_1 < R_2$, such that for all $b \in B$

$$\Delta_b s = \lambda s \implies \begin{cases} \Pi_b s = s & \text{if } \lambda < R_1, \\ \Pi_b s = 0 & \text{if } \lambda > R_2. \end{cases}$$

Remark 4.2. The triviality of the range of the spectral section is not in the original definition. We added this property because we will only consider spectral sections having as range a trivial vector bundle over B . Notice also that from the discreteness of the spectrum, this vector bundle must be of finite rank.

Obviously, one gets a spectral section by taking $\Pi_{M/B} = 0$ and $R_1 < R_2 < 0$, but this example is rather trivial. We would like to know about the existence of non-trivial spectral sections. The following result, which was communicated to the author in a private discussion, is due to R.B. Melrose.

Proposition 4.3. Let $\Delta_{M/B}$ be a family of Laplacians as in (4.1). Then given $R > 0$, there exists $R' > R$ and a smooth family of projections $\Pi_{M/B}$ such that $(\Pi_{M/B}, R, R')$ is a spectral section for $\Delta_{M/B}$.

Proof. By compactness, the discreteness of the spectrum and its continuity as a set-valued function, we can find a covering of B by a finite number of open sets Ω_i , and $R_i \in (R, \infty)$ such that R_i is not in the spectrum of Δ_b for all $b \in \Omega_i$. The span of the eigensections with eigenvalues less than R_i is then a smooth vector bundle E_i over Ω_i . Moreover, there are smooth bundle inclusions on all non-trivial intersections $\Omega_{ij} = \Omega_i \cap \Omega_j$, $I_{ij} : E_i \rightarrow E_j$, provided $R_i \leq R_j$. These inclusions satisfy obvious compatibility conditions on triple intersections.

By Kuiper's theorem (see [8]), the Hilbert bundle $L^2(M/B; E)$ is trivial in the uniform topology. It follows that there are trivial finite dimensional subbundles of arbitrary large rank. In other words, there is a sequence of families of finite rank projections $\pi_b^{(k)}$ with ranges trivial vector bundles over B such that $\pi_b^{(k)} \rightarrow \text{Id}$ in the strong topology (on operators). Setting $\pi_b = \pi_b^{(k)}$ for sufficiently large k , it follows that the restriction of π_b to the E_i are injective

$$\pi_b : E_{i,b} \hookrightarrow F_b,$$

where the trivial vector bundle F is the range of π . By taking a larger k if needed, we can also assume that for all E_i , the norm of $(\text{Id} - \pi_b)|_{E_i}$ is less than $\frac{1}{2}$. These embeddings of the bundles E_i as subbundles of $F|_{\Omega_i}$ are consistent with the inclusions I_{ij} . We may also consider the generalized inverse of π_b on E_i , $m_i : F \rightarrow E_i$ over Ω_i , defined as the composite of the orthogonal projection on $\pi(E_i)$ and the inverse of π as a map from E_i to $\pi(E_i)$.

Let μ_i be a partition of unity subordinate to the Ω_i , and consider the family of linear maps

$$(4.3) \quad f_b : F_b \rightarrow L^2(M_b; E_b), \quad f_b = (\text{Id} - \pi_b) \sum_j \mu_j(b) m_j(b).$$

This is a well-defined smooth family. Consider the open sets

$$\mathcal{U}_i = \Omega_i \setminus \bigcup_{\{j; R_j < R_i\}} \text{supp}(\mu_j).$$

These cover B . Over \mathcal{U}_i , the sum in (4.3) is limited to those j with $\Omega_{ij} \neq \emptyset$ and $m_j(\pi_b e) = e$ for $e \in E_{i,b}$. Therefore, $f_b(\pi_b e) = (\text{Id} - \pi_b)e$ for $b \in \mathcal{U}_i$ and $e \in E_{i,b}$. Since $\|f_b\| < \frac{1}{2}$, it follows that

$$(\text{Id} + f_b) : F_b \rightarrow L^2(M_b; E_b)$$

embeds F as a subbundle of $L^2(M/B; E)$ which, over \mathcal{U}_i , contains E_i as a subbundle.

Finally, choose $\psi \in C^\infty(\mathbb{R})$ with $0 \leq \psi(t) \leq 1$ and such that

$$\psi(t) = \begin{cases} 1 & \text{if } t < 0 \\ 0 & \text{if } t > 1. \end{cases}$$

For $T \in \mathbb{R}$, consider the family of linear maps $Q_T = \psi(\Delta_{M/B} - T)$. For large T , $Q_T \circ (\text{Id} + f) : F \rightarrow G$ embeds F as a spectrally finite subbundle G containing the E_i over \mathcal{U}_i . Then, if $\Pi_{M/B}$ is the orthogonal projection on G and $R' = T + 1$, we see that $(\Pi_{M/B}, R, R')$ is the desired spectral section. Since F is a trivial vector bundle, it is clear that G is a trivial vector bundle as well. \square

5. CONSTRUCTION OF THE K -CLASS

We are now ready to construct a K -class out of the indicial family of a Fredholm operator in $\mathcal{F}_\Phi^{-\infty}(X; E)$. We need to distinguish two cases, the case where $\dim Z > 0$ and the case where $\dim Z = 0$.

5.1. The case where $\dim Z > 0$. Let

$$(5.1) \quad \begin{array}{ccc} G^{-\infty}(Z; E) & \longrightarrow & G^{-\infty} \\ & & \downarrow \phi \\ & & Y \end{array}$$

be the smooth bundle with fibre at $y \in Y$ given by $G^{-\infty}(Z_y; E_y)$, where

$$(5.2) \quad G^{-\infty}(Z_y; E_y) = \{\text{Id} + Q \mid Q \in \Psi^{-\infty}(Z_y; E_y), \text{Id} + Q \text{ is invertible}\}$$

is the group of invertible smoothing perturbations of the identity discussed in the introduction and $Z_y = \Phi^{-1}(y)$, $E_y = E|_{\Phi^{-1}(y)}$. The bundle $G^{-\infty}$ is a subbundle of the bundle $\mathcal{P}^{-\infty}$ in (1.10). Let $\pi : V \rightarrow Y$ be a real vector bundle over \mathbb{R} . In many of the situations interesting us, the vector bundle V will be ${}^\Phi N^*Y$. Let $\pi^*G^{-\infty}$ denote the pull-back of $G^{-\infty}$ on V .

Definition 5.1. *A smooth section of $\pi^*G^{-\infty}$, or more generally of any bundle which has a section Id , is said to be **asymptotic to the identity** if it converges rapidly with all derivatives to Id as one approaches infinity in V . Let $\Gamma_{\text{Id}}(V; \pi^*G^{-\infty})$ denote the space of smooth sections of $\pi^*G^{-\infty}$ asymptotic to the identity.*

The following corollary is an immediate consequence of proposition 1.14 and proposition 1.17.

Corollary 5.2. *For $V = {}^\Phi N^*Y$, the indicial family $\text{Id} + \hat{A}_0$ of a Fredholm operator in $\mathcal{F}_{\Phi}^{-\infty}(X; E)$ corresponds to a section of the bundle $\pi^*G^{-\infty} \rightarrow {}^\Phi N^*Y$ which is asymptotic to the identity.*

Our primary goal in this section is to construct a K -class out of the **homotopy class** of the indicial family of a Fredholm operator $(\text{Id} + A) \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$. But considering an arbitrary real vector bundle V over Y will allow us to get a more general result which will turn out to be useful in section 7. So we will consider the more general situation of a smooth section $\text{Id} + S$ of $\pi^*G^{-\infty} \rightarrow V$ asymptotic to the identity, bearing in mind that in the case where $V = {}^\Phi N^*Y$, the section $\text{Id} + S$ can be seen as the indicial family of a Fredholm operator in $\mathcal{F}_{\Phi}^{-\infty}(X; E)$.

We want to describe the homotopy classes of such sections. To this end, choose a family of metrics $g_{\partial X/Y}$ for the fibration $\Phi : \partial X \rightarrow Y$ and a Euclidean metric g^E as well as a connection ∇^E for the bundle E . Then, as discussed in section 4, there is an associated family of Laplacians $\Delta_{\partial X/Y}$. Given $y \in Y$, recall from the introduction that if $\{f_i\}_{i \in \mathbb{N}}$ is an orthonormal basis of eigensections of Δ_y with increasing eigenvalues, then

$$(5.3) \quad \begin{aligned} F_{ij} : G^{-\infty}(Z_y; E_y) &\rightarrow \mathcal{G}^{-\infty} \\ \text{Id} + Q &\mapsto \delta_{ij} + \langle f_i, Q f_j \rangle_{L^2} \end{aligned}$$

is an isomorphism between $G^{-\infty}(Z_y; E_y)$ and the group $\mathcal{G}^{-\infty}$ of invertible semi-infinite matrices $\delta_{ij} + A_{ij}$ such that

$$(5.4) \quad \sup_{i,j} (i+j)^k |A_{ij}| < \infty, \quad \forall k \in \mathbb{N}_0,$$

where δ_{ij} corresponds to the semi-infinite identity matrix. If $P_y(T)$ denotes the projection onto the span of the eigensections of Δ_y with eigenvalues less than T , this means that

$$(5.5) \quad P_y(T)(Q_y)P_y(T) \xrightarrow{T \rightarrow +\infty} Q$$

in the \mathcal{C}^∞ -topology, so we can approximate Q by spectrally finite smoothing operators.

Lemma 5.3. *Let $(\text{Id} + S)$ be a smooth section of $\pi^*G^{-\infty} \rightarrow V$ asymptotic to the identity, where $V \rightarrow Y$ is a real vector bundle over Y . Then there exists a spectral section (Π, R_1, R_2) for $\Delta_{\partial X/Y}$ such that*

$$\|S_b - \Pi_b S_b \Pi_b\| < \frac{1}{2\|(\text{Id} + S_b)^{-1}\|}, \quad \forall b \in V,$$

where $\|\cdot\|$ is the operator norm and Π_b denotes $\Pi_{\pi(b)}$.

Proof. Using (5.5) and the fact $\text{Id} + S$ is asymptotic to the identity, we see by the compactness of Y , the discreteness of the spectrum and its continuity as a set-valued function, that there exist a covering of Y by a finite number of open sets Ω_i , and numbers $T_i \in \mathbb{R}^+$ such that

$$(5.6) \quad \|S_b - P_b(T_i) S_b P_b(T_i)\| < \frac{1}{2\|(\text{Id} + S_b)^{-1}\|}, \quad \forall b \in \pi^*\Omega_i,$$

where $P_b(T_i)$ denotes $P_{\pi(b)}(T_i)$. Set $R_1 = \max_i \{T_i\}$. Then, by proposition 4.3, there exists a smooth family of projections Π and $R_2 > R_1$ such that (Π, R_1, R_2) is a spectral section for $\Delta_{\partial X/Y}$. By construction, we have that for $b \in \pi^*\Omega_i$,

$$\Pi_b P_b(T_i) \Pi_b = P_b(T_i),$$

so the estimate (5.6) also holds for Π_b for all $b \in V$. Thus (Π, R_1, R_2) is the desired spectral section. \square

Corollary 5.4. *Let (Π, R_1, R_2) be the spectral section of lemma 5.3. Then, $\text{Id} + S$ and $\text{Id} + \Pi S \Pi$ are homotopic as sections of $\pi^*G^{-\infty}$ asymptotic to the identity.*

Proof. Consider the smooth homotopy

$$(5.7) \quad t \mapsto \text{Id} + S + t(\Pi S \Pi - S), \quad t \in [0, 1].$$

For all $b \in V$ and $t \in [0, 1]$, notice that

$$(\text{Id} + S_b)^{-1}(\text{Id} + S_b + t(\Pi_b S_b \Pi_b - S_b)) = \text{Id} + t(\text{Id} + S_b)^{-1}(\Pi_b S_b \Pi_b - S_b)$$

is invertible, since $\|t(\text{Id} + S_b)^{-1}(\Pi_b S_b \Pi_b - S_b)\| < \frac{1}{2}$. This means that

$$\text{Id} + S_b + t(\Pi_b S_b \Pi_b - S_b)$$

is invertible as well, and so the homotopy (5.7) between $(\text{Id} + S)$ and $(\text{Id} + \Pi S \Pi)$ is through sections of $\pi^*G^{-\infty}$ asymptotic to the identity. \square

Let F denote the range of Π . Recall by our definition of a spectral section that F is a trivial vector bundle of finite rank over V . Let $\text{GL}(F, \mathbb{C}) \rightarrow \Phi N^*Y$ be the smooth bundle over V with fibre at $b \in V$ given by $\text{GL}(F_{\pi(b)}, \mathbb{C})$, the group of complex linear isomorphism of $F_{\pi(b)}$. Then corollary 5.4 effectively reduces the section $\text{Id} + S$ to a section

$$(5.8) \quad \text{Id} + S_\Pi = \Pi(\text{Id} + S)\Pi : V \rightarrow \text{GL}(F, \mathbb{C}).$$

If we also choose an explicit trivialization of F and if n is the rank of F , then the section $\text{Id} + S$ can be seen as smooth map

$$(5.9) \quad \text{Id} + S_\Pi : V \rightarrow \text{GL}(n, \mathbb{C})$$

asymptotic to the identity. Finally, since $\text{GL}(n, \mathbb{C})$ is a topological subspace of the direct limit

$$(5.10) \quad \text{GL}(\infty, \mathbb{C}) = \lim_{k \rightarrow \infty} \text{GL}(k, \mathbb{C}),$$

we can think of (5.9) as a map

$$(5.11) \quad \text{Id} + S_\Pi : V \rightarrow \text{GL}(\infty, \mathbb{C})$$

which converges to the identity at infinity. We are mostly interested in the homotopy class of this map.

Definition 5.5. Let $[V ; \text{GL}(\infty, \mathbb{C})]$ denote the homotopy classes of continuous maps from V to $\text{GL}(\infty, \mathbb{C})$ which converges to the identity as one approaches infinity in V . If a is such a continuous map, then let $[a]$ denote its homotopy class in $[V ; \text{GL}(\infty, \mathbb{C})]$.

Remark 5.6. The passage from the category of smooth maps to the category of continuous maps does not change the set of homotopy classes, see for instance proposition 17.8 in [5].

Definition 5.7. If $(\text{Id} + S)$ and (Π, R_1, R_2) are as in corollary 5.4, then let $[\text{Id} + S]_\infty$ denote the homotopy class of (5.11) in $[V ; \text{GL}(\infty, \mathbb{C})]$.

The homotopy class $[\text{Id} + S]_\infty$ depends a priori on three choices:

- (1) The choice of an explicit trivialization of F ,
- (2) The choice of a spectral section as in corollary 5.4,
- (3) The choice of a family of Laplacians $\Delta_{\partial X/Y}$.

In the next three lemmas, we will show that in fact it is independent of these three choices.

Lemma 5.8. For $\Delta_{\partial X/Y}$ and (Π, R_1, R_2) fixed, the homotopy class $[\text{Id} + S]_\infty$ does not depend on the way F is trivialized.

Proof. Suppose $(\text{Id} + S_\Pi) : V \rightarrow \text{GL}(\infty, \mathbb{C})$ and $(\text{Id} + S'_\Pi) : V \rightarrow \text{GL}(\infty, \mathbb{C})$ arise from two different trivializations of F . Then,

$$\text{Id} + S'_\Pi = M(\text{Id} + S_\Pi)M^{-1}$$

for some smooth map $M : V \rightarrow \text{GL}(\infty, \mathbb{C})$ which is a pull-back of a map from Y to $\text{GL}(\infty, \mathbb{C})$. But at the homotopy level, the product $[a] \circ [b] = [ab]$ is commutative (see for instance lemma 2.4.6 in [2]), so

$$\begin{aligned} [\text{Id} + S'_\Pi] &= [M(\text{Id} + S_\Pi)M^{-1}] = [M] \circ [\text{Id} + S_\Pi] \circ [M^{-1}] \\ &= [\text{Id} + S_\Pi] \circ [M] \circ [M^{-1}] = [\text{Id} + S_\Pi] \circ [\text{Id}] \\ &= [\text{Id} + S_\Pi]. \end{aligned}$$

□

Lemma 5.9. Assume the family of Laplacians $\Delta_{\partial X/Y}$ is fixed and let $(\text{Id} + S)$ be as in lemma 5.3. If (Π_1, R_1, T_1) and (Π_2, R_2, T_2) are two spectral sections as in lemma 5.3, then $\text{Id} + S_{\Pi_1}$ and $\text{Id} + S_{\Pi_2}$ define the same homotopy class in $[V ; \text{GL}(\infty, \mathbb{C})]$.

Proof. For $k \in \mathbb{N}$, set $R'_k = \max\{T_1, T_2\} + k$. Then proposition 4.3 asserts the existence of a sequence of spectral sections $\{(\Pi'_k, R'_k, T'_k)\}_{k \in \mathbb{N}}$ for the family of Laplacians $\Delta_{\partial X/Y}$ on Y . By construction,

$$\Pi'_k \Pi_i \Pi'_k = \Pi_i \quad \text{for } i \in \{1, 2\}, k \in \mathbb{N},$$

so the estimate of lemma 5.3 applies as well to all the projections Π'_k :

$$\|S_b - \Pi'_{k,b} S_b \Pi'_{k,b}\| < \frac{1}{2\|(\text{Id} + S_b)^{-1}\|}, \quad \forall b \in V, \quad \forall k \in \mathbb{N}.$$

Moreover, since $R'_k \xrightarrow[k \rightarrow \infty]{} \infty$, we know from (5.5), the fact that $\text{Id} + S$ is asymptotic to the identity and the compactness of $Y \times [0, 1]$ that in the \mathcal{C}^∞ -topology,

$$\Pi'_{k,b}(\text{Id} + S_b + t(\Pi_{i,b} S_b \Pi_{i,b} - S_b)) \Pi'_{k,b} + \text{Id} - \Pi'_{k,b} \xrightarrow[k \rightarrow \infty]{} \text{Id} + S_b + t(\Pi_{i,b} S_b \Pi_{i,b} - S_b)$$

uniformly in $(b, t) \in V \times [0, 1]$, $i \in \{1, 2\}$. By the proof of corollary 5.4, the limits of these two sequences are homotopies of sections of $\pi^* G^{-\infty}$ asymptotic to the identity. Thus, taking $k \in \mathbb{N}$ large enough, we can assume that

$$t \mapsto \Pi'_{k,b}(\text{Id} + S_b + t(\Pi_{i,b} S_b \Pi_{i,b} - S_b)) \Pi'_{k,b} + \text{Id} - \Pi'_{k,b}, \quad t \in [0, 1],$$

are homotopies of sections of $\pi^* G^{-\infty}$ asymptotic to the identity for $i \in \{1, 2\}$. Then, if F_i and F'_k denote the range of Π_i and Π'_k , we see that

$$(5.12) \quad t \mapsto \Pi'_k(\text{Id} + S + t(\Pi_i S \Pi_i - S)) \Pi'_k, \quad t \in [0, 1],$$

is a homotopy of sections of $\text{GL}(F'_k, \mathbb{C})$ between $\Pi'_k(\text{Id} + S) \Pi'_k$ and $\Pi'_k(\text{Id} + \Pi_i S \Pi_i) \Pi'_k$ for $i \in \{1, 2\}$. It is possible that the complements of the F_i in F'_k are not trivial bundles, but by considering a spectral section (Π_j, R'_j, T''_j) with $R'_j > T'_k$ and j large enough, we can add a trivial bundle to F'_k so that the complements of the F_i become trivial¹ and so that (Π_k, R_k, T''_j) is still a spectral section. So without loss of generality, we can assume that the complements of the F_i in F'_k are trivial. Then, we see from lemma 5.8 that

$$\begin{aligned} [\text{Id} + S_{\Pi_1}] &= [\text{Id} + (\Pi_1 S \Pi_1) \Pi'_k] \\ &= [\text{Id} + S_{\Pi'_k}] \quad \text{using (5.12) with } i = 1, \\ &= [\text{Id} + (\Pi_2 S \Pi_2) \Pi'_k] \quad \text{using (5.12) with } i = 2, \\ &= [\text{Id} + S_{\Pi_2}]. \end{aligned}$$

□

Lemma 5.10. *The homotopy class $[\text{Id} + S]_\infty$ of definition 5.7 does not depend on the choice of the family of Laplacians $\Delta_{\partial X/Y}$.*

Proof. Let Δ_0 and Δ_1 be two different families of Laplacians for the fibration $\Phi : \partial X \rightarrow Y$ and the complex vector bundle E , arising respectively from the metrics $g_0^{T(\partial X/Y)}, g_0^E$ and $g_1^{T(\partial X/Y)}, g_1^E$, and connections ∇_0^E and ∇_1^E . We need to show that they lead to the same homotopy class in $[V ; \text{GL}(\infty, \mathbb{C})]$.

By considering the homotopies of metrics

$$t \mapsto g_t^{T(\partial X/Y)} = (1 - t)g_0^{T(\partial X/Y)} + t g_1^{T(\partial X/Y)}, \quad t \in [0, 1],$$

and

$$t \mapsto g_t^E = (1 - t)g_0^E + t g_1^E, \quad t \in [0, 1],$$

and the homotopy of connections

$$t \mapsto (1 - t)\nabla_0^E + t\nabla_1^E, \quad t \in [0, 1],$$

one gets an associated homotopy of families of Laplacians

$$t \mapsto \Delta_t, \quad t \in [0, 1],$$

¹For instance, we can add a trivial bundle isomorphic to $F_1 \oplus F_2$.

where Δ_t is defined using the metrics $g_t^{T(\partial X/Y)}$ and g_t^E and the connection ∇_t^E . We can think of Δ_t as a family of Laplacians parametrized by $Y \times [0, 1]$. A moment of reflection reveals that the proofs of lemma 5.3 and corollary 5.4 apply equally well if Y is replaced by $Y \times [0, 1]_t$ and $\text{Id} + S$ is replaced by the section $\text{Id} + p^*S$, where $p : V \times [0, 1]_t \rightarrow V$ is the projection on the left factor. This leads to a spectral section (Π, R_1, R_2) satisfying the estimate of lemma 5.3 and such that

$$t \mapsto \text{Id} + p^*S + t(\Pi(p^*S)\Pi - p^*S), \quad t \in [0, 1]$$

is a homotopy of sections of $p^*\pi^*G^{-\infty}$ between $(\text{Id} + p^*S)$ and $(\text{Id} + \Pi(p^*S)\Pi)$. If Π_t denotes the restriction of Π to the slice $Y \times \{t\}$, then (Π_0, R_1, R_2) and (Π_1, R_1, R_2) are spectral sections for Δ_0 and Δ_1 which satisfy the conclusions of lemma 5.3 and corollary 5.4. Moreover, using lemma 5.8, we see from the homotopy

$$t \mapsto \text{Id} + \Pi_t S \Pi_t, \quad t \in [0, 1],$$

that $[\text{Id} + S_{\Pi_0}]_{\infty} = [\text{Id} + S_{\Pi_1}]_{\infty}$. From lemma 5.9, we conclude that the homotopy class of definition 5.7 is the same for Δ_0 and Δ_1 . \square

From lemmas 5.8, 5.9 and 5.10, we get the following

Proposition 5.11. *The homotopy class $[\text{Id} + S]_{\infty}$ of definition 5.7 is well-defined, that is, it is independent of the three choices involved in its construction.*

Definition 5.12. *Let $[V ; \pi^*G^{-\infty}]$ denote the set of homotopy classes of smooth sections of $\pi^*G^{-\infty}$ asymptotic to the identity. If $\text{Id} + S$ is a section of $\pi^*G^{-\infty}$ asymptotic to the identity, let $[\text{Id} + S]$ denotes its homotopy class in $[V ; \pi^*G^{-\infty}]$.*

Lemma 5.13. *If $\text{Id} + S$ is a section of $\pi^*G^{-\infty}$ asymptotic to the identity, then $[\text{Id} + S]_{\infty}$ in $[V ; \text{GL}(\infty, \mathbb{C})]$ only depends on the homotopy class of $\text{Id} + S$ in $[V ; \pi^*G^{-\infty}]$.*

Proof. Assume that $t \mapsto \text{Id} + S_t$, $t \in [0, 1]$, is a homotopy of sections of $\pi^*G^{-\infty}$ asymptotic to the identity. We need to show that $[\text{Id} + S_0]_{\infty} = [\text{Id} + S_1]_{\infty}$. Since $[0, 1]$ is compact, a moment of thought reveals that the proofs of lemma 5.3 and corollary 5.4 work equally well if one replaces V by the bundle $[0, 1] \times V \rightarrow Y$. What one gets is that there exists a spectral section (Π, R_1, R_2) for $\Delta_{\partial X/Y}$ on Y so that

$$\|S_t(b) - \Pi_b S_t(b) \Pi_b\| < \frac{1}{2\|(\text{Id} + S_t(b))^{-1}\|}, \quad \forall b \in V, t \in [0, 1],$$

which implies that $\text{Id} + S_t$ and $\text{Id} + \Pi S_t \Pi$ are homotopic as smooth homotopies of smooth sections of $\pi^*G^{-\infty}$ asymptotic to the identity. In particular, after trivializing the range of Π , $\text{Id} + \Pi S_0 \Pi$ and $\text{Id} + \Pi S_1 \Pi$ define the same homotopy class in $[V ; \text{GL}(\infty, \mathbb{C})]$, and we conclude that

$$[\text{Id} + S_0]_{\infty} = [\text{Id} + S_1]_{\infty}.$$

\square

Thus, we see that definition 5.7 gives us a canonical map

$$(5.13) \quad \begin{array}{ccc} I_{\infty} : & [V ; \pi^*G^{-\infty}] & \rightarrow [V ; \text{GL}(\infty, \mathbb{C})] \\ & [\text{Id} + S] & \mapsto [\text{Id} + S]_{\infty} . \end{array}$$

Proposition 5.14. *The canonical map I_{∞} is an isomorphism of sets.*

Proof. The proof of the surjectivity of I_∞ reduces to the existence of a spectral section with range a vector bundle of arbitrary large rank. But the existence of such a spectral section is an easy consequence of proposition 4.3 together with the compactness of Y and the continuity of the spectrum as a set-valued function.

For the proof of the injectivity of I_∞ , suppose $[\text{Id} + S_0]_\infty = [\text{Id} + S_1]_\infty$. Let Π_0 and Π_1 be spectral sections as in lemma 5.3 that can be used to define $[\text{Id} + S_0]_\infty$ and $[\text{Id} + S_1]_\infty$ respectively. Let Π be another spectral section such that $\Pi \Pi_i \Pi = \Pi_i$ for $i \in \{0, 1\}$ (cf. proof of lemma 5.9). Then Π can also be used to define $[\text{Id} + S_0]_\infty$ and $[\text{Id} + S_1]_\infty$. Taking Π to have a larger rank if necessary, we can assume that there is a homotopy

$$t \mapsto \text{Id} + \Pi S_t \Pi, \quad t \in [0, 1],$$

through sections of $\pi^* G^{-\infty}$ asymptotic to the identity. From corollary 5.4, we then deduce that $[\text{Id} + S_0] = [\text{Id} + S_1]$ in $[V ; \pi^* G^{-\infty}]$. This shows that the map I_∞ is injective. \square

Before discussing the associated K -class, let us concentrate on the second case.

5.2. The case where $\dim Z = 0$. The case where $\dim Z = 0$ turns out to be significantly easier. This is somehow the reverse procedure of the case where $\dim Z > 0$, that is, instead of reducing an infinite bundle to a trivial vector bundle, we will enlarge a finite vector bundle to make it trivial. The case $\dim Z = 0$ evidently includes scattering operators, but it means more generally that the fibration $\Phi : \partial X \rightarrow Y$ is a finite covering.

Definition 5.15. Let $\mathcal{E} \rightarrow Y$ be the complex vector bundle on Y with fibre at $y \in Y$ given by

$$\mathcal{E}_y = \bigoplus_{z \in \Phi^{-1}(y)} E_y.$$

Definition 5.16. Let $\text{GL}(\mathcal{E}, \mathbb{C})$ be the bundle over Y with fibre at $y \in Y$ given by $\text{GL}(\mathcal{E}_y, \mathbb{C})$. Let $\pi^* \text{GL}(\mathcal{E}, \mathbb{C}) \rightarrow {}^\Phi N^* Y$ be the pull-back of $\text{GL}(\mathcal{E}, \mathbb{C})$ on ${}^\Phi N^* Y$.

An immediate consequence of remark 1.16 and proposition 1.14 is the following

Corollary 5.17. The indicial family $(\text{Id} + \hat{A}_0)$ of a Fredholm operator $(\text{Id} + A) \in \mathcal{F}_\Phi^{-\infty}(X; E)$ is a section of the bundle $\pi^* \text{GL}(\mathcal{E}; \mathbb{C})$ which is asymptotic to the identity.

Let $\mathcal{F} \rightarrow Y$ be a complex vector bundle over Y such that $\mathcal{E} \oplus \mathcal{F}$ is trivial. Such a bundle always exists, see for instance corollary 1.4.14 in [2]. If n is the rank of $\mathcal{E} \oplus \mathcal{F}$, so that $\mathcal{E} \oplus \mathcal{F} \cong \underline{\mathbb{C}}^n$, then

$$\begin{aligned} \pi^* \text{GL}(\mathcal{E}, \mathbb{C}) &\subset \pi^* \text{GL}(\mathcal{E} \oplus \mathcal{F}, \mathbb{C}) \\ &\cong \pi^* \text{GL}(\underline{\mathbb{C}}^n, \mathbb{C}) \\ (5.14) \quad &= \text{GL}(n, \mathbb{C}) \times {}^\Phi N^* Y \\ &\subset \text{GL}(\infty, \mathbb{C}) \times {}^\Phi N^* Y. \end{aligned}$$

Definition 5.18. If $(\text{Id} + S)$ is the indicial family of some operator in $\mathcal{F}_\Phi^{-\infty}(X; E)$, let $(\text{Id} + S_\mathcal{F}) : {}^\Phi N^* Y \rightarrow \text{GL}(\infty; \mathbb{C})$ be the associated map under the series of inclusions (5.14) and let $[\text{Id} + S]_\infty \in [{}^\Phi N^* Y ; \text{GL}(\infty; \mathbb{C})]$ denotes its homotopy class.

As in the previous case, we want to show that the homotopy class $[\text{Id} + S]_\infty$ does not depend on the choices made to define it, namely the choice of an explicit trivialization of $\mathcal{E} \oplus \mathcal{F}$ and the choice of \mathcal{F} .

Proposition 5.19. *The homotopy class $[\text{Id} + S]_\infty$ of definition 5.18 is well-defined, that is, it does not depend on the choice of \mathcal{F} and on the way $\mathcal{E} \oplus \mathcal{F}$ is trivialized.*

Proof. First, the homotopy class $[\text{Id} + S]_\infty$ does not depend on the way $\mathcal{E} \oplus \mathcal{F}$ is trivialized. It is the same proof as lemma 5.8.

Secondly, the homotopy class $[\text{Id} + S]_\infty$ does not depend on the choice of the vector bundle \mathcal{F} such that $\mathcal{E} \oplus \mathcal{F}$ is trivial.

Indeed, let $\mathcal{G} \rightarrow Y$ be another complex vector bundle such that $\mathcal{E} \oplus \mathcal{G}$ is trivial. Then consider the trivial vector bundle $\mathcal{H} = \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{E}' \oplus \mathcal{G}$, where $\mathcal{E}' \cong \mathcal{E}$. The bundles $\mathcal{E} \oplus \mathcal{F}$ and $\mathcal{E} \oplus \mathcal{G}$ are subbundles of \mathcal{H} , and to be more precise, we consider the inclusions

$$\begin{array}{ccc} i_1 : \mathcal{E} \oplus \mathcal{F} & \hookrightarrow & \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{E}' \oplus \mathcal{G} \\ (e, f) & \mapsto & (e, f, 0, 0) \end{array} \quad , \quad \begin{array}{ccc} i_2 : \mathcal{E} \oplus \mathcal{G} & \hookrightarrow & \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{E}' \oplus \mathcal{G} \\ (e, g) & \mapsto & (e, 0, 0, g) \end{array} .$$

Clearly, the complements of $\mathcal{E} \oplus \mathcal{F}$ and $\mathcal{E} \oplus \mathcal{G}$ in \mathcal{H} are trivial vector bundles, so this means that \mathcal{F} and $\mathcal{F} \oplus \mathcal{E}' \oplus \mathcal{G}$ lead to the same homotopy class, and similarly for \mathcal{G} and $\mathcal{F} \oplus \mathcal{E}' \oplus \mathcal{G}$. In particular \mathcal{F} and \mathcal{G} lead to same homotopy class. \square

In this case, it is straightforward to see that the homotopy class $[\text{Id} + S]_\infty$ only depends on the homotopy class of $\text{Id} + S$ in $[\Phi N^* Y ; \pi^* \text{GL}(\mathcal{E}, \mathbb{C})]$. So definition 5.18 gives also rise to a canonical map

$$(5.15) \quad \begin{array}{ccc} I_\infty : [\Phi N^* Y ; \pi^* \text{GL}(\mathcal{E}, \mathbb{C})] & \rightarrow & [\Phi N^* Y ; \text{GL}(\infty, \mathbb{C})] \\ [\text{Id} + S] & \mapsto & [\text{Id} + S]_\infty . \end{array}$$

However, it is not an isomorphism in general, but all classes arise this way if arbitrary bundles are admitted.

5.3. The associated K -class. Now that in both cases $\dim Z > 0$ and $\dim Z = 0$ we can associate a well-defined homotopy class

$$[\text{Id} + \hat{A}_0]_\infty \in [\Phi N^* Y ; \text{GL}(\infty; \mathbb{C})]$$

to the indicial family $(\text{Id} + \hat{A}_0)$ of a Fredholm operator $(\text{Id} + A)$ in $\mathcal{F}_\Phi^{-\infty}(X; E)$, let us try to reinterpret this homotopy class in terms of K -theory. We will follow the notation of [2]. Recall that the space $\text{GL}(\infty, \mathbb{C})$ is a classifying space for odd K -theory², so we have the correspondence

$$[\Phi N^* Y ; \text{GL}(\infty, \mathbb{C})] \cong \tilde{K}^0(\mathbb{S}^1 \wedge Y^{\Phi N^* Y}),$$

where $Y^{\Phi N^* Y}$ is the **Thom space** of $\Phi N^* Y$, in other words its one point compactification, and

$$\mathbb{S}^1 \wedge Y^{\Phi N^* Y} = (\mathbb{S}^1 \times Y^{\Phi N^* Y}) / (\mathbb{S}^1 \times \{\infty\} \cup \{1\} \times Y^{\Phi N^* Y})$$

is the **reduced suspension** of $Y^{\Phi N^* Y}$, $\mathbb{S}^1 \subset \mathbb{C}$ being the unit circle. Since $\Phi N^* Y \cong T^* Y \times \mathbb{R}$, we see that

$$Y^{\Phi N^* Y} \cong \mathbb{S}^1 \wedge Y^{T^* Y} = (\mathbb{S}^1 \times Y^{T^* Y}) / (\mathbb{S}^1 \times \{\infty\} \cup \{1\} \times Y^{T^* Y}),$$

²See for instance lemma 2.4.6 in [2]

the factor “ \mathbb{R} ” being effectively turned into a suspension. This means that

$$\begin{aligned}
 [\Phi N^*Y; \text{GL}(\infty, \mathbb{C})] &\cong \tilde{K}^0(\mathbb{S}^1 \wedge \mathbb{S}^1 \wedge Y^{T^*Y}) \\
 &= \tilde{K}^{-2}(Y^{T^*Y}) \\
 (5.16) \quad &\cong \tilde{K}^0(Y^{T^*Y}) \quad \text{by the periodicity theorem} \\
 &= K_c^0(T^*Y),
 \end{aligned}$$

where $K_c^0(T^*Y)$, the **K -theory with compact support** of T^*Y , is by definition equal to $\tilde{K}^0(Y^{T^*Y})$. Thus, we see from (5.16) that the homotopy class $[\text{Id} + \hat{A}_0]$ in $[\Phi N^*Y; \text{GL}(\infty, \mathbb{C})]$ defines a K -class in $K_c^0(T^*Y)$.

Definition 5.20. *If $(\text{Id} + \hat{A}_0)$ is the indicial family of a Fredholm operator $(\text{Id} + A)$ in $\mathcal{F}_\Phi^{-\infty}(X; E)$, let $\kappa(\text{Id} + A) \in K_c^0(T^*Y)$ denote the K -class associated to the homotopy class $[\text{Id} + \hat{A}_0]_\infty$ under the identification $[\Phi N^*Y; \text{GL}(\infty, \mathbb{C})] \cong K_c^0(T^*Y)$ given by (5.16).*

6. THE INDEX IN TERMS OF K -THEORY

Having associated a K -class $\kappa(\text{Id} + A) \in K_c^0(T^*Y)$ to a Fredholm operator $(\text{Id} + A)$ in $\mathcal{F}_\Phi^{-\infty}(X; E)$, the obvious guess is that the index of $\text{Id} + A$ is given by

$$(6.1) \quad \text{ind}(\text{Id} + A) = \text{ind}_t(\kappa(\text{Id} + A))$$

where

$$\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z}$$

is the topological index introduced by Atiyah and Singer in [3]. This is indeed the case. We will prove (6.1) by reducing the computation of the index to the case of a scattering operator.

To the manifold X with fibred cusp structure given by the fibration $\Phi : \partial X \rightarrow Y$ and some defining function x of ∂X , let us associate the manifold with boundaries

$$(6.2) \quad \mathcal{W}_Y = [0, 1]_t \times Y, \quad \partial \mathcal{W}_Y = Y_0 \cup Y_1,$$

where $Y_0 = \{0\} \times Y \subset \mathcal{W}_Y$ and $Y_1 = \{1\} \times Y \subset \mathcal{W}_Y$. The boundary $\partial \mathcal{W}_Y$ has an obvious defining function given by t near Y_0 and by $(1 - t)$ near Y_1 , where $t \in [0, 1]$ is the variable of the left factor of $[0, 1]_t \times Y$. It will have a fibred cusp structure if one considers the trivial fibration given by the identity

$$\text{Id} : \partial \mathcal{W}_Y \rightarrow \partial \mathcal{W}_Y.$$

If $F \rightarrow \mathcal{W}_Y$ is a complex vector bundle on \mathcal{W}_Y , then we can define the algebra of fibred cusp operators $\Psi_{\text{Id}}^*(\mathcal{W}_Y; F)$ acting on sections of F . From the previous section, we know that a Fredholm operator in $\mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ gives rise to a well define element in

$$K_c^0(T^*(\partial \mathcal{W}_Y)) \cong K_c^0(T^*Y_0) \oplus K_c^0(T^*Y_1).$$

Definition 6.1. *Let $i_1 : K_c^0(T^*Y) \hookrightarrow K_c^0(T^*(\partial \mathcal{W}_Y))$ be the natural inclusion given by*

$$\begin{aligned}
 i_1 : K_c^0(T^*Y) &\hookrightarrow K_c^0(T^*Y_0) \oplus K_c^0(T^*Y_1) \\
 a &\mapsto (0, a).
 \end{aligned}$$

Lemma 6.2. *If $(\text{Id} + A) \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ is such that the indicial family \widehat{A}_0 , seen as a section of $\pi^* \mathcal{P}^{-\infty}$, lies in some subbundle $\pi^* \text{End}(F) \subset \pi^* \mathcal{P}^{-\infty}$, where $F \rightarrow Y$ is some trivial finite rank complex vector bundle, then any scattering operator $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ such that*

$$\kappa(\text{Id} + S) = i_1 \kappa(\text{Id} + A), \text{ in } K_c^0(T^* \mathcal{W}_Y)$$

has the same index as $\text{Id} + A$.

Proof. By proposition 3.3, we can assume that $N'_{\Phi}(A) = \widehat{A}_0$, that is, $\widehat{A}_k = 0$ for $k \in \mathbb{N}$. From proposition 3.7 and formula (3.5), we see that a parametrix $\text{Id} + B$ of $\text{Id} + A$ with full normal operator

$$\widehat{N'_{\Phi}(B)} = \sum_{k=0}^{\infty} x^k \widehat{B}_k$$

is such that \widehat{B}_k is a section of $\pi^* \text{End}(F) \subset \pi^* \mathcal{P}^{-\infty}$. According to proposition 3.2, the index of $\text{Id} + A$ is then given by

$$(6.3) \quad \text{ind}(\text{Id} + A) = \frac{1}{(2\pi)^{l+1}} \int_{\Phi N^* Y} \text{Tr}_{\Phi^{-1}(y)}((D_{\log x} \widehat{AB})_{l+1}(y, \tau, \eta)) dy d\eta d\tau.$$

But from lemma 3.8,

$$N'_{\Phi}(\widehat{D_{\log x} A}) = \sum_{k=1}^{\infty} \frac{x^k D_{\tau}^k \widehat{A}_0}{k}.$$

Thus, since $\widehat{B}_k, D_{\tau}^k \widehat{A}_0$ are sections of $\pi^* \text{End}(F) \subset \pi^* \mathcal{P}^{-\infty}$ for all $k \in \mathbb{N}_0$, this means the trace $\text{Tr}_{\Phi^{-1}(y)}$ in (6.3) can be replaced by the trace on F_y ,

$$(6.4) \quad \text{ind}(\text{Id} + A) = \frac{1}{(2\pi)^{l+1}} \int_{\Phi N^* Y} \text{Tr}_{F_y}((D_{\log x} \widehat{AB})_{l+1}(y, \tau, \eta)) dy d\eta d\tau.$$

But applying proposition 3.2 to scattering operators, we see from proposition 3.7 that (6.4) also gives the index of an operator $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ such that $\widehat{N'_{\text{Id}}(S)} = 0$ on $\text{Id} N^* Y_0$ and $\widehat{N'_{\text{Id}}(S)} = \widehat{A}_0$ on $\text{Id} N^* Y_1$. Clearly, $\kappa(\text{Id} + S) = i_1 \kappa(\text{Id} + A)$. More generally, from formula (6.4), definition 5.18 and proposition 3.2, any $(\text{Id} + Q) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ such that $\kappa(\text{Id} + Q) = i_1 \kappa(\text{Id} + A)$ has the same index as $\text{Id} + S$ and $\text{Id} + A$. \square

Lemma 6.3. *Let $(\text{Id} + A) \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ be a Fredholm operator and let $\kappa(\text{Id} + A)$ be its associated K -class in $K_c^0(T^* Y)$. Then, for F a trivial vector bundle with sufficiently large rank, there exists a scattering operator $\text{Id} + S \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ with associated K -class given by $i_1 \kappa(\text{Id} + A) \in K_c^0(T^*(\partial \mathcal{W}_Y))$ and such that $\text{ind}(\text{Id} + A) = \text{ind}(\text{Id} + S)$.*

Proof. The proof relies on lemma 6.2, but is slightly different depending on whether or not $\dim Z > 0$.

First assume that $\dim Z > 0$. Then let (Π, R_1, R_2) be a spectral section as in lemma 5.3 for the indicial family $\text{Id} + \widehat{A}_0$. By corollary 5.4, $(\text{Id} + \Pi \widehat{A}_0 \Pi)$ is homotopic to $(\text{Id} + \widehat{A}_0)$. Let $P \in \Psi_{\Phi}^{-\infty}(X; E)$ be such that $\widehat{P}_0 = \Pi \widehat{A}_0 \Pi$. Then by proposition 1.14 and proposition 3.3, $(\text{Id} + P) \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ has the same index as $\text{Id} + A$ and it defines the same K -class in $K_c^0(T^* Y)$. By taking $F \rightarrow Y$ to be the range of Π on Y , we can then apply lemma 6.2 to $\text{Id} + P$. Thus there exists $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ such that $\text{ind}(\text{Id} + S) = \text{ind}(\text{Id} + P)$ and $\kappa(\text{Id} + S) = i_1 \kappa(\text{Id} + P)$. So $\text{Id} + S$ is the desired operator.

If $\dim Z = 0$ let $\mathcal{E} \rightarrow Y$ be as in definition 5.15. From lemma 6.2, there exists $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; \mathcal{E})$ having the same index as $\text{Id} + A$ and having $i_1 \kappa(\text{Id} + A)$ as an associated K -class. If \mathcal{E} is not trivial, let $\mathcal{G} \rightarrow \mathcal{W}_Y$ be another complex vector bundle such that $F = \mathcal{E} \oplus \mathcal{G}$ is trivial and let $\text{Id} + S' \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ be the operator which acts as $(\text{Id} + S)$ on sections of \mathcal{E} and as the identity on sections of \mathcal{G} . Clearly, $\text{Id} + S'$ has the same index as $\text{Id} + S$ and they define the same K -class. So $\text{Id} + S'$ is the desired operator. \square

This reduces the problem to the computation of the index of a scattering operator. In section 6.5 of [15], a general topological formula for the index of scattering operators is derived. For the convenience of the reader, we will adapt the discussion that can be found there to our particular context.

Given $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; F)$ as in lemma 6.3 with $F = \mathbb{C}^n$, the indicial family $\text{Id} + \widehat{S}_0$ of $\text{Id} + S$ can be seen as a map

$$(6.5) \quad \text{Id} + \widehat{S}_0 : {}^{\text{Id}}N^* \partial \mathcal{W}_Y \rightarrow \text{GL}(n, \mathbb{C}).$$

The homotopy class of this map gives rise to the associated K -class in $K_c^0(T^* \partial \mathcal{W}_Y)$ under the identification (5.16). Let \mathcal{V}_Y be the double of \mathcal{W}_Y obtained by taking two copies \mathcal{W}_Y and \mathcal{W}'_Y of \mathcal{W}_Y and identifying Y_i of the first copy with Y'_i of the second copy for $i \in \{0, 1\}$. Thus, $\mathcal{V}_Y \cong \mathbb{S}^1 \times Y$.

Inside \mathcal{V}_Y , ${}^{\text{Id}}N^* \partial \mathcal{W}_Y$ naturally identifies with the conormal bundle $N^*(T^* \partial \mathcal{W}_Y)$ of $T^* \partial \mathcal{W}_Y$. Via the clutching construction (see p.75 in [2]), the map (6.5) defines a K -class with compact support in $K_c^0(T^* \mathcal{V}_Y)$. More generally,

Definition 6.4. For $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; \mathbb{C}^n)$, let $c(\text{Id} + S) \in K_c^0(T^* \mathcal{V}_Y)$ be the element defined by applying the clutching construction to the map (6.5).

The following proposition is a particular case of theorem 6.4 in [15].

Proposition 6.5. The index of $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; \mathbb{C}^n)$ is given by

$$\text{ind}(\text{Id} + S) = \text{ind}_t(c(\text{Id} + S))$$

where $\text{ind}_t : K_c^0(T^* \mathcal{V}_Y) \rightarrow \mathbb{Z}$ is the topological index map.

Proof. Let us enlarge our space of operators and see $\text{Id} + S$ as a Fredholm operator in $\Psi_{\text{Id}}^0(\mathcal{W}_Y; \mathbb{C}^n)$. Then proposition 1.14 can be reinterpreted as saying that the K -class defined by the symbol $\sigma_0(\text{Id} + S)$ is null when restricted to $T^* \mathcal{W}_Y|_{\partial \mathcal{W}_Y}$. This means one can smoothly deform $\text{Id} + S$ through Fredholm operators in $\Psi_{\text{Id}}^0(\mathcal{W}_Y; \mathbb{C}^n)$ to a Fredholm operator $P \in \Psi_{\text{Id}}^0(\mathcal{W}_Y; \mathbb{C}^n)$ that acts by multiplication of a matrix near $\partial \mathcal{W}_Y$. This new operator is no longer in $\mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; \mathbb{C}^n)$ in general, but it has the same index as $\text{Id} + S$. A general Fredholm operator $Q \in \Psi_{\text{Id}}^0(\mathcal{W}_Y; \mathbb{C}^n)$ also defines an element of $K_c^0(T^* \mathcal{V}_Y)$ but one needs also the symbol. If $\overline{T^* \mathcal{W}_Y}$ is the radial compactification of $T^* \mathcal{W}_Y$, then the indicial family \widehat{Q}_0 together with the symbol σ_0 form a continuous map

$$f : \partial(\overline{T^* \mathcal{W}_Y}) \rightarrow \text{GL}(n, \mathbb{C}).$$

This map can be used to define a relative K -class on the double $D_{\overline{T^* \mathcal{W}_Y}}$ of $\overline{T^* \mathcal{W}_Y}$, obtained by identifying two copies of $\overline{T^* \mathcal{W}_Y}$ at their boundaries. This class will in fact have support inside $T^* \mathcal{V}_Y \subset D_{\overline{T^* \mathcal{W}_Y}}$. Thus this relative K -class

is really an element of $K_c^0(T^*\mathcal{V}_Y)$. When $Q \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; \underline{\mathbb{C}}^n)$, it is not hard to see that this construction reduces to definition 6.4.

Coming back to P , this means that P defines the same element in $K_c^0(T^*\mathcal{V}_Y)$ as $c(\text{Id} + S)$. Now, since P acts as a matrix near the boundary, it defines a vector bundle $\mathcal{E} \rightarrow \mathcal{V}_Y$ via the clutching construction and P can be extended to act as the identity on $\mathcal{E}|_{\mathcal{W}_Y}$. This defines a pseudodifferential operator $\tilde{P} \in \Psi^0(\mathcal{V}_Y; \mathcal{E})$ having the same index as P . Moreover, it is not hard to see that the K -class in $K_c^0(T^*\mathcal{V}_Y)$ defined by the symbol of \tilde{P} is simply $c(\text{Id} + S)$. By the Atiyah-Singer index theorem, the index of \tilde{P} is then given by $\text{ind}_t(c(\text{Id} + S))$, which completes the proof. \square

The formula for the index is now easy to get.

Theorem 6.6. *The index of a Fredholm operator $(\text{Id} + A) \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ is given by*

$$\text{ind}(\text{Id} + A) = \text{ind}_t(\kappa(\text{Id} + A))$$

where $\kappa(\text{Id} + A) \in K_c^0(T^*Y)$ is the associated K -class of definition 5.20 and

$$\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z}$$

is the topological index map of Atiyah and Singer.

Proof. By lemma 6.3, for $n \in \mathbb{N}$ large enough, there exists a scattering operator $(\text{Id} + S) \in \mathcal{F}_{\text{Id}}^{-\infty}(\mathcal{W}_Y; \underline{\mathbb{C}}^n)$ which has the same index as $\text{Id} + A$ and such that $\kappa(\text{Id} + S) = i_1 \kappa(\text{Id} + A)$. By the previous proposition, the index of $\text{Id} + S$ is given by

$$\text{ind}(\text{Id} + S) = \text{ind}_t(c(\text{Id} + S)),$$

where $c(\text{Id} + S) \in K_c^0(T^*\mathcal{V}_Y)$ is defined using the clutching construction. At $Y_0 \subset \partial\mathcal{W}_Y$, the indicial family of $(\text{Id} + S)$ is trivial. This means the clutching construction only depends on the part of the indicial family of $\text{Id} + S$ defined over ${}^{\text{Id}}N^*Y_1$. Let

$$(6.6) \quad c : [{}^{\Phi}N^*Y ; \text{GL}(n; \mathbb{C})] \rightarrow K_c^0(T^*\mathcal{V}_Y)$$

denotes the map obtained by applying the clutching construction, where Y is identified with $Y_1 \subset \mathcal{V}_Y$. The identification

$$[{}^{\Phi}N^*Y ; \text{GL}(\infty; \mathbb{C})] \cong [\mathbb{R} \times T^*Y ; \text{GL}(\infty; \mathbb{C})] \cong \tilde{K}^{-2}(Y^{T^*Y})$$

combined with (6.6) gives a map

$$f : \tilde{K}^{-2}(Y^{T^*Y}) \rightarrow K_c^0(T^*\mathcal{V}_Y).$$

Let us decompose the map f into simpler maps. If we identify the tangent bundle with the cotangent bundle via some metric, T^*Y_1 can be seen as a submanifold of $T^*\mathcal{V}_Y$. Then, a tubular neighborhood of T^*Y_1 in $T^*\mathcal{V}_Y$ defines an inclusion

$$i : N(T^*Y_1) \hookrightarrow T^*\mathcal{V}_Y$$

of the normal bundle $N(T^*Y_1)$ of T^*Y_1 . This in turn defines a push-forward map in K -theory

$$i_* : K_c^0(N(T^*Y_1)) \rightarrow K_c^0(T^*\mathcal{V}_Y).$$

On the other hand, since $N(T^*Y_1) \cong \mathbb{R}^2 \times T^*Y$, we see that there is an isomorphism

$$j : \tilde{K}^{-2}(Y^{T^*Y}) \xrightarrow{\sim} K_c^0(N(T^*Y_1)).$$

It is not hard to see that the map f is given by

$$f = i_* \circ j : \tilde{K}^{-2}(Y^{T^*Y}) \rightarrow K_c^0(T^*\mathcal{V}_Y).$$

Combined with the periodicity isomorphism $p : K_c^0(T^*Y) \xrightarrow{\sim} \tilde{K}^{-2}(Y^{T^*Y})$, we see that

$$c(\text{Id} + S) = i_* \circ j \circ p(\kappa(\text{Id} + A)).$$

Now, the composition of maps $j \circ p$ is just the Thom isomorphism

$$\varphi : K_c^0(T^*Y) \rightarrow K_c^0(N(T^*Y_1))$$

applied to the trivial bundle $\underline{\mathbb{C}}$ over T^*Y , since $N(T^*Y_1) \cong \mathbb{C} \times T^*Y$. Thus, one has

$$c(\text{Id} + S) = i_! \kappa(\text{Id} + A)$$

where $i_! = i_* \varphi$. From the commutativity of the diagram³

$$\begin{array}{ccc} K_c^0(T^*Y) & \xrightarrow{i_!} & K_c^0(T^*\mathcal{V}_Y) \\ \downarrow \text{ind}_t & & \downarrow \text{ind}_t \\ \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} \end{array},$$

we conclude that

$$\begin{aligned} \text{ind}(\text{Id} + A) &= \text{ind}(\text{Id} + S) \\ &= \text{ind}_t c(\text{Id} + S) = \text{ind}_t i_! \kappa(\text{Id} + A) \\ &= \text{ind}_t \kappa(\text{Id} + A). \end{aligned}$$

□

Corollary 6.7. *When $\dim Z > 0$, the index map $\text{ind} : \mathcal{F}_\Phi^{-\infty}(X; E) \rightarrow \mathbb{Z}$ is surjective.*

Proof. From theorem 6.6, it suffices to prove that the map

$$\kappa : \mathcal{F}_\Phi^{-\infty}(X; E) \rightarrow K_c^0(T^*Y)$$

of definition 5.20 is surjective, which is equivalent to showing that any homotopy class in $[\Phi N^*Y; \text{GL}(\infty, \mathbb{C})]$ can be realized through definition 5.7 as $[\text{Id} + A]_\infty$ for some $(\text{Id} + A) \in \mathcal{F}_\Phi^{-\infty}(X; E)$.

So let $[f] \in [\Phi N^*Y; \text{GL}(\infty, \mathbb{C})]$ be arbitrary. Then, there exists $n \in \mathbb{N}$ depending on $[f]$ so that the homotopy class $[f]$ has a smooth representative

$$f : \Phi N^*Y \rightarrow \text{GL}(n, \mathbb{C}) \subset \text{GL}(\infty, \mathbb{C}).$$

Considering some family of Laplacians as in the beginning of section 5 and using proposition 4.3, there exists an associated spectral section (Π, R_1, R_2) with range a trivial complex vector bundle $\underline{\mathbb{C}}^m \rightarrow \Phi N^*Y$ with $m \geq n$. By taking n larger if needed, we can assume $m = n$. Thus taking $(\text{Id} + A) \in \mathcal{F}_\Phi^{-\infty}(X; E)$ so that

$$\begin{aligned} \Pi_b(\text{Id} + \hat{A}_0(b))\Pi_b &= f(b), \quad \forall b \in \Phi N^*Y, \\ (\text{Id} - \Pi_b)(\hat{A}_0(b))(\text{Id} - \Pi_b) &= 0, \quad \forall b \in \Phi N^*Y, \end{aligned}$$

we see that $[\text{Id} + A]_\infty = [f]$. □

Corollary 6.8. *When $\dim Z = 0$ the index map $\text{ind} : \mathcal{F}_\Phi^{-\infty}(X; \underline{\mathbb{C}}^n) \rightarrow \mathbb{Z}$ is surjective provided $n \in \mathbb{N}$ is large enough.*

³see p.501 in [3]

Proof. As in the previous corollary, it suffices to show that any homotopy class in $[\Phi N^*Y ; \mathrm{GL}(\infty, \mathbb{C})]$ can be written as $[\mathrm{Id} + A]$ for some $\mathrm{Id} + A \in \mathcal{F}_\Phi^{-\infty}(X; \underline{\mathbb{C}}^n)$. But there exists $n \in \mathbb{N}$ depending on the dimension of Y so that any homotopy class in $[\Phi N^*Y ; \mathrm{GL}(\infty, \mathbb{C})]$ can be represented by a map from ΦN^*Y to $\mathrm{GL}(n, \mathbb{C})$, and clearly such maps can always be chosen to be the indicial family of some Fredholm operator in $\mathcal{F}_\Phi^{-\infty}(X; \underline{\mathbb{C}}^n)$. \square

7. THE HOMOTOPY GROUPS OF $G_\Phi^{-\infty}$

When homotopy groups come into play, the cases where $\dim Z > 0$ and where $\dim Z = 0$ are radically different. In some sense, the case where $\dim Z > 0$ is to the case where $\dim Z = 0$ what stable homotopy groups of $\mathrm{GL}(n, \mathbb{C})$ are to the actual homotopy groups of $\mathrm{GL}(n, \mathbb{C})$. That is to say, the homotopy groups are much easier to describe in the first case. What explains this difference is that the bundle $\pi^*G^{-\infty}$ is infinite dimensional in the case where $\dim Z > 0$. This gives more freedom in the way one can perform homotopies, which has the effect of reducing the set of homotopy classes, and thus of simplifying the picture.

Still, it is possible to say something in the case where $\dim Z = 0$ if one consents to allow some stabilization. This goes as follows. Let X be a compact manifold with boundary ∂X together with a fibred cusp structure given by a finite covering $\Phi : \partial X \rightarrow Y$. Then consider the manifold with boundary $\mathbb{S}^1 \times X$ with fibred cusp structure given by

$$\begin{aligned} \Phi_{\mathbb{S}^1} : \mathbb{S}^1 \times X &\rightarrow Y \\ (s, p) &\mapsto \Phi(p) . \end{aligned}$$

Let $E \rightarrow X$ be any complex vector bundle over X . Then E can be seen as a subbundle of the (trivial) Hilbert bundle $L^2(\mathbb{S}^1)$ over X , and Fredholm operators in $\mathcal{F}_\Phi^{-\infty}(X; E)$ can be extended by letting them act as the identity on the complement of E in $L^2(\mathbb{S}^1) \rightarrow X$. This gives an embedding

$$\mathcal{F}_\Phi^{-\infty}(X; E) \subset \mathcal{F}_{\Phi_{\mathbb{S}^1}}^{-\infty}(\mathbb{S}^1 \times X; \underline{\mathbb{C}}).$$

In this way, the space $\mathcal{F}_{\Phi_{\mathbb{S}^1}}^{-\infty}(\mathbb{S}^1 \times X; \underline{\mathbb{C}})$ is a stabilization of $\mathcal{F}_\Phi^{-\infty}(X; E)$ which brings us back to the case where $\dim Z > 0$. To the reader not completely satisfied by this compromise, let us point out where our argument breaks down when $\dim Z = 0$. It is in the failure of being able to prove lemma 7.17, which is an important ingredient in the proof of the surjectivity of the boundary homomorphism occurring in the long exact sequence (7.2).

From now on, we will assume that $\dim Z > 0$.

Definition 7.1. Let $G_\Phi^{-\infty}(X; E) \subset \mathcal{F}_\Phi^{-\infty}(X; E)$ be the group of invertible operators in $\mathcal{F}_\Phi^{-\infty}(X; E)$, that is,

$$G_\Phi^{-\infty}(X; E) = \{\mathrm{Id} + A \mid A \in \Psi_\Phi^{-\infty}(X; E) \text{ and } (\mathrm{Id} + A) \text{ is invertible}\}.$$

We want to compute the homotopy groups of $G_\Phi^{-\infty}(X; E)$. Our approach will be similar to the one in [17] in the sense that we will compute the homotopy groups of $G_\Phi^{-\infty}(X; E)$ out of a long exact sequence of homotopy groups. To this end, let us introduce the other spaces involved.

Definition 7.2. Let $\dot{G}^{-\infty}(X; E)$ be the group of operators given by

$$\dot{G}^{-\infty}(X; E) = \{\text{Id} + Q \mid Q \in x^\infty \Psi_\Phi^{-\infty}(X; E), \text{Id} + Q \text{ is invertible}\}.$$

Remark 7.3. By considering the eigensections vanishing in Taylor series at the boundary of a Laplacian arising from a fibred cusp metric (see (8.1) in [11]), one can identify $\dot{G}^{-\infty}(X; E)$ with $\mathcal{G}^{-\infty}$, the group of semi-infinite invertible matrices satisfying (5.4). This means $\dot{G}^{-\infty}(X; E)$ is a classifying space for odd K -theory. Notice also that $\dot{G}^{-\infty}(X; E)$ does not depend on the choice of a fibred cusp structure.

Definition 7.4. Let $G_{\Phi_s}^{-\infty}(\partial X; E)$ denote the group

$$\{\text{Id} + Q \mid Q \in \Psi_{\Phi_s}^{-\infty}(\partial X; E) \text{ and } \text{Id} + Q \text{ is invertible}\},$$

that is, the group of invertible ${}^\Phi NY$ -suspended smoothing perturbations of the identity. Taking the Fourier transform in ${}^\Phi NY$, this group naturally identifies with the group $\Gamma_{\text{Id}}({}^\Phi N^*Y; \pi^*G^{-\infty})$ of smooth sections of $\pi^*G^{-\infty}$ asymptotic to the identity. This is the point of view we will adopt.

Lemma 7.5. The homotopy groups of $G_{\Phi_s}^{-\infty}(\partial X; E)$ are:

$$\pi_k(G_{\Phi_s}^{-\infty}(\partial X; E)) \cong \begin{cases} K_c^0(T^*Y) & , \quad k \text{ even}, \\ \tilde{K}^{-1}(Y^{T^*Y}) & , \quad k \text{ odd}. \end{cases}$$

Proof. By proposition 5.14, we have

$$\begin{aligned} \pi_k(G_{\Phi_s}^{-\infty}(\partial X; E)) &= \pi_k(\Gamma_{\text{Id}}({}^\Phi N^*Y; \pi^*G^{-\infty})) \\ &\cong [{}^\Phi N^*Y \times \mathbb{R}^k; \pi^*G^{-\infty}] \\ &\cong [T^*Y \times \mathbb{R}^{k+1}; \text{GL}(\infty, \mathbb{C})] \\ &\cong \tilde{K}^{-1}(\mathbb{S}^{k+1} \wedge Y^{T^*Y}) = \tilde{K}^{-k-2}(Y^{T^*Y}). \end{aligned}$$

The result then follows from the periodicity theorem of K -theory. \square

Proposition 7.6. The homotopy groups of $\mathcal{F}_\Phi^{-\infty}(X; E)$ are the same as those of $G_{\Phi_s}^{-\infty}(\partial X; E)$, namely

$$\pi_k(\mathcal{F}_\Phi^{-\infty}(X; E)) \cong \begin{cases} K_c^0(T^*Y) & , \quad k \text{ even}, \\ \tilde{K}^{-1}(Y^{T^*Y}) & , \quad k \text{ odd}. \end{cases}$$

Proof. Consider the short exact sequence

$$\text{Id} + x\Psi_\Phi^{-\infty}(X; E) \hookrightarrow \mathcal{F}_\Phi^{-\infty}(X; E) \xrightarrow{N_\Phi} G_{\Phi_s}^{-\infty}(\partial X; E).$$

This is a Serre fibration. One can see this using Seeley extension for manifolds with corners and thinking in terms of Schwartz kernels. The space $\text{Id} + x\Psi_\Phi^{-\infty}(X; E)$ is obviously contractible. So we deduce from the associated long exact sequence of homotopy groups that for $k \in \mathbb{N}$,

$$\pi_k(\mathcal{F}_\Phi^{-\infty}(X; E)) \cong \pi_k(G_{\Phi_s}^{-\infty}(\partial X; E)).$$

That $\pi_0(\mathcal{F}_\Phi^{-\infty}(X; E)) \cong \pi_0(G_{\Phi_s}^{-\infty}(\partial X; E))$ is clear from Seeley extension for manifolds with corners and the contractibility of $(\text{Id} + x\Psi_\Phi^{-\infty}(X; E))$. \square

Definition 7.7. Let $\mathcal{F}_{\Phi,0}^{-\infty}(X;E)$ denote the subspace of Fredholm operators in $\mathcal{F}_{\Phi}^{-\infty}(X;E)$ having index zero. Let $G_{\Phi s,0}^{-\infty}(\partial X;E)$ be the subgroup of $G_{\Phi s}^{-\infty}(\partial X;E)$ given by

$$G_{\Phi s,0}^{-\infty}(\partial X;E) = N_{\Phi}(\widehat{\mathcal{F}_{\Phi,0}^{-\infty}(X;E)}),$$

and let $G_{\Phi s,0}^{-\infty}(X;E)[[x]]$ denotes the group of operators

$$G_{\Phi s,0}^{-\infty}(X;E)[[x]] = N'_{\Phi}(\widehat{\mathcal{F}_{\Phi,0}^{-\infty}(X;E)})$$

where the composition law is given by the $*$ -product of definition 3.6.

Remark 7.8. Notice that $G_{\Phi s,0}^{-\infty}(\partial X;E)$ has the same homotopy groups as the space $G_{\Phi s}^{-\infty}(\partial X;E)$ but fewer connected components, more precisely,

$$\pi_0(G_{\Phi s,0}^{-\infty}(\partial X;E)) \cong \pi_0(\mathcal{F}_{\Phi,0}^{-\infty}(X;E)) \cong \ker [\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z}].$$

Moreover, keeping only the term of order x^0 gives a deformation retraction of $G_{\Phi s,0}^{-\infty}(X;E)[[x]]$ onto $G_{\Phi s,0}^{-\infty}(X;E)$, so $G_{\Phi s,0}^{-\infty}(X;E)[[x]]$ as the same homotopy groups and the same set of connected components as $G_{\Phi s,0}^{-\infty}(X;E)$.

Lemma 7.9. The full normal operator N'_{Φ} gives rise to a short exact sequence

$$(7.1) \quad \dot{G}^{-\infty}(X;E) \hookrightarrow G_{\Phi}^{-\infty}(X;E) \xrightarrow{N'_{\Phi}} G_{\Phi s,0}^{-\infty}(X;E)[[x]]$$

which is a Serre fibration.

Proof. The injectivity and the exactness in the middle are clear. For the surjectivity, let $(\text{Id}+Q) \in G_{\Phi s,0}^{-\infty}(X;E)[[x]]$ be arbitrary, and let $(\text{Id}+P) \in \mathcal{F}_{\Phi,0}^{-\infty}(X;E)$ be such that $\widehat{N'_{\Phi}(P)} = Q$. By proposition 1.15, the kernel and the cokernel of $\text{Id}+P$ are contained in $\dot{\mathcal{C}}^{\infty}(X;E)$. Since $\text{Id}+P$ has index zero, there exists a linear isomorphism $\varphi : \ker P \rightarrow \text{coker } P$. By extending it to act as 0 on the orthogonal complement of $\ker P$ in $L^2(X;E)$, it can be seen as an element of $x^{\infty}\Psi_{\Phi}^{-\infty}(X;E)$, so that

$$\text{Id}+P+\varphi \in G_{\Phi}^{-\infty}(X;E), \quad \widehat{N'_{\Phi}(\text{Id}+P+\varphi)} = Q.$$

This shows that N'_{Φ} is surjective. The proof that (7.1) is a Serre fibration is basically the same as the proof of lemma 3.5 in [17]. \square

A Serre fibration has an associated long exact sequence of homotopy groups, so we have the following

Corollary 7.10. The Serre fibration (7.1) has an associated long exact sequence of homotopy groups

$$(7.2) \quad \begin{aligned} \cdots \xrightarrow{\partial} \pi_k(\dot{G}^{-\infty}) &\longrightarrow \pi_k(G_{\Phi}^{-\infty}) \longrightarrow \pi_k(G_{\Phi s,0}^{-\infty}) \xrightarrow{\partial} \pi_{k-1}(\dot{G}^{-\infty}) \longrightarrow \cdots \\ &\cdots \xrightarrow{\partial} \pi_0(\dot{G}^{-\infty}) \longrightarrow \pi_0(G_{\Phi}^{-\infty}) \longrightarrow \pi_0(G_{\Phi s,0}^{-\infty}), \end{aligned}$$

where ∂ is the boundary homomorphism and $\dot{G}^{-\infty}$, $G_{\Phi}^{-\infty}$ and $G_{\Phi s,0}^{-\infty}$ denote respectively $\dot{G}^{-\infty}(X;E)$, $G_{\Phi}^{-\infty}(X;E)$ and $G_{\Phi s,0}^{-\infty}(\partial X;E)[[x]]$.

Since we know the homotopy groups of $\dot{G}^{-\infty}(X;E)$ and $G_{\Phi s,0}^{-\infty}(X;E)$, we will be able to compute the homotopy groups of $G_{\Phi}^{-\infty}(X;E)$ provided we identify the boundary homomorphism ∂ . In fact, it is only necessary to know that ∂ is surjective. But even with a complete knowledge of ∂ , there would still be an ambiguity

concerning the connected components of $G_{\Phi}^{-\infty}(X; E)$. Let us get rid of this ambiguity immediately.

Lemma 7.11. *The set of connected components of $G_{\Phi}^{-\infty}(X; E)$ is given by*

$$\pi_0(G_{\Phi}^{-\infty}(X; E)) \cong \ker [\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z}].$$

Proof. From proposition 7.6 and theorem 6.6, it suffices to prove that

$$(7.3) \quad \pi_0(G_{\Phi}^{-\infty}(X; E)) \longrightarrow \pi_0(\mathcal{F}_{\Phi}^{-\infty}(X; E)) \xrightarrow{\text{ind}} \mathbb{Z}$$

is a short exact sequence. The surjectivity follows from corollary 6.7. The exactness in the middle follows using proposition 1.15 (cf. the proof of the surjectivity in lemma 7.9). Finally, the injectivity follows from the long exact sequence (7.2), the fact that $\pi_0(\dot{G}^{-\infty}(X; E)) = \{0\}$ and proposition 7.6. \square

To prove that the boundary homomorphism is surjective, we will see it as a generalization of the index map $\text{ind} : \pi_0(\mathcal{F}_{\Phi}^{-\infty}(X; E)) \rightarrow \mathbb{Z}$ to all homotopy groups of $\mathcal{F}_{\Phi}^{-\infty}(X; E)$. Let us recall from the appendix in [2] how can one performs such a generalization.

Definition 7.12. *If M is a compact manifold with basepoint m_0 , we denote by $[M ; \mathcal{F}_{\Phi}^{-\infty}(X; E)]$ the set of homotopy classes of continuous maps from M to $\mathcal{F}_{\Phi}^{-\infty}(X; E)$ which send the basepoint m_0 to the identity.*

We will define an index map

$$(7.4) \quad \text{ind} : [M ; \mathcal{F}_{\Phi}^{-\infty}(X; E)] \rightarrow K^0(M),$$

where $\tilde{K}^0(M)$ is the reduced even K -theory of M . Let H denote the Hilbert space $L^2(X; E)$ defined using some smooth positive density on X . Given a continuous map

$$\begin{array}{ccc} T : & M & \rightarrow \mathcal{F}_{\Phi}^{-\infty}(X; E) \\ & m & \mapsto T_m, \end{array}$$

with $T_{m_0} = \text{Id}$, one can show⁴ the existence of a vector space $V \subset H$ which is closed of finite codimension such that $V \cap \ker T_m = \{0\}$ for all $m \in M$, and such that

$$\bigcup_{m \in M} H/T_m(V),$$

topologized as a quotient space of $M \times H$, is a vector bundle over M . Let us denote this vector bundle by $H/T(V)$.

Definition 7.13. *Let $T : M \rightarrow \mathcal{F}_{\Phi}^{-\infty}(X; E)$ and $V \subset H$ be as above. Then the **index** of T is defined to be*

$$\text{ind}(T) = [H/V] - [H/T(V)] \in \tilde{K}^0(M).$$

⁴see proposition A5 in [2]

One can check that the index of T only depends on the homotopy class of T and that it does not depend on the choice of $V \subset H$. Moreover, if $S : M \rightarrow \mathcal{F}_\Phi^{-\infty}(X; E)$ is another continuous map, then

$$(7.5) \quad \text{ind}(TS) = \text{ind}(T) + \text{ind}(S).$$

For later convenience, we will give some precision about the way one can choose the vector space V used in definition 7.13. First, notice that the orthogonal complement V^\perp of V in H is naturally isomorphic to H/V . Moreover, the vector bundle $T(V)^\perp \rightarrow M$ with fibre at $m \in M$ given by the orthogonal complement $T_m(V)^\perp$ of $T_m(V)$ is naturally isomorphic to the vector bundle $H/T(V)$. Hence, the index can also be written

$$\text{ind}(T) = [V^\perp] - [T(V)^\perp].$$

Lemma 7.14. *It is possible to choose the vector space V occuring in definition 7.13 such that*

$$V^\perp \subset \dot{\mathcal{C}}^\infty(X; E), \quad T_m(V)^\perp \subset \dot{\mathcal{C}}^\infty(X; E) \quad \forall m \in M.$$

Moreover, if $\text{ind}(T) = 0$, we can also choose V so that the vector bundles V^\perp and $T(V)^\perp$ are isomorphic.

Proof. According to the proof of proposition A5 in [2], one can always choose V to be of the form

$$V = \bigcap_{i=1}^n (\ker T_{m_i})^\perp \Rightarrow V^\perp = \sum_{i=1}^n \ker T_{m_i},$$

for some finite number of points $m_1, \dots, m_n \in M$. Thus, by proposition 1.15, $V^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$. Moreover, for all $m \in M$,

$$(7.6) \quad v \in T_m(V)^\perp \Rightarrow v \in \ker T_m^* + T_m(V^\perp),$$

so by proposition 1.15, this means $T_m(V)^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$.

If $\text{ind}(T) = 0$, then there exists a trivial bundle P over M so that

$$H/V \oplus P \cong H/T(V) \oplus P.$$

By taking a subspace $W \subset V$ such that $\dim(V/W) = \text{rank}(P)$, we have

$$H/W \cong H/T(W).$$

Without loss of generality, we can assume as well that $W^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$, which implies as in (7.6) that $T_m(W)^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$ for all $m \in M$. \square

We know from proposition 7.6 that the normal operator N_Φ induces an isomorphism

$$(7.7) \quad N_{\Phi*} : \pi_k(\mathcal{F}_\Phi^{-\infty}(X; E)) \rightarrow \pi_k(G_{\Phi_s}^{-\infty}(X; E)) = \pi_k(G_{\Phi_s, 0}^{-\infty}(X; E)[[x]]), \quad k \in \mathbb{N}.$$

Moreover, since $\dot{G}^{-\infty}(X; E)$ is a classifying space for odd K -theory, we get an isomorphism via the clutching construction

$$(7.8) \quad c : \pi_k(\dot{G}^{-\infty}(X; E)) \rightarrow \tilde{K}^{-2}(\mathbb{S}^{k-1}) = \tilde{K}^0(\mathbb{S}^{k+1}), \quad k \in \mathbb{N}.$$

Proposition 7.15. *For $k \in \mathbb{N}$, the homomorphism*

$$c \circ \partial \circ N_{\Phi*} : \pi_k(\mathcal{F}_\Phi^{-\infty}(X; E)) \rightarrow \tilde{K}^0(\mathbb{S}^k)$$

is the index map of definition 7.13 with $M = \mathbb{S}^k$.

Proof. Let $T : \mathbb{S}^k \rightarrow \mathcal{F}_\Phi^{-\infty}(X; E)$ be a representative of an element of the homotopy group $\pi_k(\mathcal{F}_\Phi^{-\infty}(X; E))$. We assume that $T_{s_0} = \text{Id}$, where $s_0 \in \mathbb{S}^k$ is the basepoint of \mathbb{S}^k . By deforming T if needed, we can assume that $T \equiv \text{Id}$ in an open ball $B_0^k \subset \mathbb{S}^k$ containing s_0 . Let the closed ball \overline{B}_1^k be the complement of B_0^k in \mathbb{S}^k .

Let $V \subset H$ be a closed subspace of H of finite codimension that can be used to define the index of T :

$$\text{ind}(T) = [H/V] - [H/T(V)] = [V^\perp] - [T(V)^\perp].$$

By lemma 7.14, we can assume that $V^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$ and $T_s(V)^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$ for all $s \in \mathbb{S}^k$. When we restrict T to the closed ball $\overline{B}_1^k \subset \mathbb{S}^k$, the index become zero because \overline{B}_1^k is contractible and $T \equiv \text{Id}$ on $\partial\overline{B}_1^k$. Thus, by lemma 7.14, we can also assume that V is such that V^\perp and $T(V)^\perp$ are isomorphic vector bundles when restricted to \overline{B}_1^k . Let $\varphi : V^\perp \rightarrow T(V)^\perp$ be a (smooth) isomorphism between V^\perp and $T(V)^\perp$ on \overline{B}_1^k . By extending φ to act by 0 on V , we get an associated family of bounded operator $\phi : \overline{B}_1^k \rightarrow \mathcal{L}(H, H)$. Since $V^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$ and $T_s(V)^\perp \subset \dot{\mathcal{C}}^\infty(X; E)$ for all $s \in \mathbb{S}^k$, this is in fact a map

$$\phi : \overline{B}_1^k \rightarrow x^\infty \Psi_\Phi^{-\infty}(X; E).$$

By construction of ϕ and the compactness of \overline{B}_1^k , we see that for $\lambda > 0$ large enough,

$$T_s + \lambda\phi(s) \in G_\Phi^{-\infty}(X; E), \quad \forall s \in \overline{B}_1^k.$$

Thus, rescaling ϕ if needed, we can assume $T + \phi$ is a map of the form

$$T + \phi : \overline{B}_1^k \rightarrow G_\Phi^{-\infty}(X; E).$$

Notice that $N'_\Phi \circ (T + \phi) = N'_\Phi \circ T$, so $T + \phi$ is in fact a lift of the map

$$N'_\Phi \circ T : \overline{B}_1^k \rightarrow G_{\Phi, s, 0}^{-\infty}(X; E)[[x]].$$

Moreover, since $T \equiv \text{Id}$ on $\partial\overline{B}_1^k \cong \mathbb{S}^{k-1}$, the map $T + \phi$ takes value in $\dot{G}^{-\infty}(X; E)$ when restricted to $\partial\overline{B}_1^k$. By definition of the boundary homomorphism (see for instance section 17.1 in [20]),

$$\partial([N'_\Phi \circ T]) = [(T + \phi)|_{\partial\overline{B}_1^k}] \in \pi_{k-1}(\dot{G}^{-\infty}(X; E)).$$

Now, since $V^\perp = T(V)^\perp$ canonically on $\partial\overline{B}_1^k$, ϕ is just a map

$$\phi : \partial\overline{B}_1^k \rightarrow \text{End}(V^\perp, V^\perp).$$

Clearly, the clutching construction applied to $(\text{Id} + \phi)^{-1}$ gives the virtual bundle $[T(V)^\perp] - [V^\perp]$, which means that the clutching construction applied to $(\text{Id} + \phi)$ gives $[V^\perp] - [T(V)^\perp]$. This shows that $c\partial N_{\Phi*}(T) = \text{ind}(T)$, which concludes the proof. \square

Since (7.7) and (7.8) are isomorphisms, the surjectivity of the boundary homomorphism is equivalent to the surjectivity of the index map of definition 7.13. In [2], there is a proof of the surjectivity of the index map, but considering the space $\mathcal{F}(H)$ of all Fredholm operators acting on H . It is possible to adapt this proof to our situation so that it still works. We first need to discuss further the topology of the space $\mathcal{F}_\Phi^{-\infty}(X; E)$.

For $n \in \mathbb{N}$, consider the vector bundle $E^n = E \otimes \mathbb{C}^n$. The bundle E can be seen as a subbundle of E^n via the inclusion

$$\begin{aligned} i : E &\rightarrow E \otimes \mathbb{C}^n \\ e &\mapsto e \otimes (1, \underbrace{0, \dots, 0}_{n-1 \text{ times}}). \end{aligned}$$

This gives an inclusion

$$(7.9) \quad \begin{aligned} i' : \mathcal{F}_{\Phi}^{-\infty}(X; E) &\hookrightarrow \mathcal{F}_{\Phi}^{-\infty}(X; E^n) \\ \text{Id} + A &\mapsto i'(\text{Id} + A), \end{aligned}$$

where $i'(\text{Id} + A)$ acts on sections of $E \subset E^n$ as $(\text{Id} + A)$ do and acts on sections of the complement of E in E^n as the identity.

Definition 7.16. A subspace \mathcal{U} of a topological space \mathcal{T} is said to be a **weak deformation retract** of \mathcal{T} if for any closed manifold M and any continuous map $f : M \rightarrow \mathcal{T}$, there exists a continuous map g homotopic to f such that $g(M) \subset \mathcal{U}$.

Lemma 7.17. Under the inclusion (7.9), $\mathcal{F}_{\Phi}^{-\infty}(X; E)$ is a weak deformation retract of $\mathcal{F}_{\Phi}^{-\infty}(X; E^n)$.

Proof. Let M be a closed manifold and let $f : M \rightarrow \mathcal{F}_{\Phi}^{-\infty}(X; E^n)$ be a continuous map. Without loss of generality, we can assume f is smooth. Compose it with the normal operator N_{Φ} to get a map

$$\tilde{f} = N_{\Phi} \circ f : M \rightarrow G_{\Phi_s}^{-\infty}(\partial X; E^n).$$

Let $\Delta_{\partial X/Y}^E$ be a family of Laplacians associated to the fibration $\Phi : \partial X \rightarrow Y$ and the complex vector bundle E (not E^n). By the compactness of M , applying an argument similar to the proof of lemma 5.3, we see that there exists a spectral section (Π, R_1, R_2) for $\Delta_{\partial X/Y}^E$ such that $(\Pi^n = \bigoplus_{i=1}^n \Pi, R_1, nR_2)$, which is a spectral section for $\bigoplus_{i=1}^n \Delta_{\partial X/Y}^E$, satisfies the estimate

$$\|\tilde{f}_b(m) - \text{Id} - \Pi_b^n(\tilde{f}_b(m) - \text{Id})\Pi_b^n\| < \frac{1}{2\|\tilde{f}_b(m)^{-1}\|}, \quad \forall b \in {}^{\Phi}N^*Y, m \in M.$$

By the proof of corollary 5.4, this means that

$$t \mapsto \tilde{f}_t = \tilde{f} + t(\Pi^n(\tilde{f})\Pi^n - (\tilde{f} - \text{Id})), \quad t \in [0, 1],$$

is a homotopy of smooth maps from M to $G_{\Phi_s}^{-\infty}(\partial X; E)$.

Let F be the range of Π . Taking Π to have range with a larger rank if necessary, we can assume that \tilde{f}_1 is homotopic, through smooth maps from M to $\Gamma_{\text{Id}}({}^{\Phi}N^*Y; \pi^* \text{GL}(F^n, \mathbb{C}))$, to a map \tilde{f}_2 with

$$\tilde{f}_2(M) \subset \Gamma_{\text{Id}}({}^{\Phi}N^*Y; \pi^* \text{GL}(F, \mathbb{C})),$$

where $\text{GL}(F, \mathbb{C}) \subset \text{GL}(F^n, \mathbb{C})$ acts as the identity on the complement of F in F^n . In particular, this means that

$$\tilde{f}_2(M) \subset G_{\Phi_s}^{-\infty}(\partial X; E) \subset G_{\Phi_s}^{-\infty}(\partial X; E^n),$$

where the inclusion $G_{\Phi_s}^{-\infty}(\partial X; E) \subset G_{\Phi_s}^{-\infty}(\partial X; E^n)$ is induced from the inclusion (7.9). Let $f_2 : M \rightarrow \mathcal{F}_{\Phi}^{-\infty}(X; E)$ be a map such that $\widehat{N_{\Phi}(f_2)} = \tilde{f}_2$, which is possible by Seeley extension for manifold with corners. Then, using again Seeley extension, one can lift the homotopy between $\tilde{f} = \tilde{f}_0$ and \tilde{f}_2 to a homotopy between f and f_2 through smooth maps from M to $\mathcal{F}_{\Phi}^{-\infty}(X; E^n)$.

□

Proposition 7.18. *For $k \in \mathbb{N}$, the boundary homomorphism*

$$\partial : \pi_k(G_{\Phi s, 0}^{-\infty}(X; E)[[x]]) \rightarrow \pi_{k-1}(\dot{G}^{-\infty}(X; E))$$

is surjective.

Proof. By proposition 7.15, it suffices to show that the index map

$$\text{ind} : \pi_k(\mathcal{F}_{\Phi}^{-\infty}(X; E)) \rightarrow \tilde{K}^0(\mathbb{S}^k)$$

is surjective.

Let $\mathcal{E} \rightarrow \mathbb{S}^k$ be an arbitrary complex vector bundle over \mathbb{S}^k . Let \mathcal{F} be another complex vector bundle such that $\mathcal{E} \oplus \mathcal{F} = \underline{\mathbb{C}}^n$ is trivial. For $s \in \mathbb{S}^k$, let $p_s \in \text{End}(\mathbb{C}^n)$ be the projection onto $\mathcal{E}_s \subset \mathbb{C}^n$.

Let $T_{-1} \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ be an operator such that $\text{ind}(T_{-1}) = -1$. Such an operator exists by corollary 6.7. Consider the family of operators

$$\tilde{T}_s = T_{-1} \otimes p_s + \text{Id} \otimes (1 - p_s) \in \mathcal{F}_{\Phi}^{-\infty}(X; E \otimes \underline{\mathbb{C}}^n), \quad s \in \mathbb{S}^k,$$

acting on $H \otimes \mathbb{C}^n$. Clearly, $\text{ind}(\tilde{T}) = -[\mathcal{E}]$. By lemma 7.17, we can deform the family \tilde{T} to a family

$$R : \mathbb{S}^k \rightarrow \mathcal{F}_{\Phi}^{-\infty}(X; E) \subset \mathcal{F}_{\Phi}^{-\infty}(X; E \otimes \underline{\mathbb{C}}^n).$$

By homotopy invariance of the index, $\text{ind}(R) = -[\mathcal{E}]$. If $\text{rank}(\mathcal{E}) = j$, let $T_j \in \mathcal{F}_{\Phi}^{-\infty}(X; E)$ be an operator of index j , then

$$\text{ind}(T_j \circ R) = \text{ind}(T_j) + \text{ind}(R) = [j] - [\mathcal{E}].$$

By construction, $T_j \circ R_s$ has index zero for all $s \in \mathbb{S}^k$. In particular, deforming $T_j \circ R$ if necessary, we can assume that $T_j \circ R_{s_0}$ is invertible at the basepoint $s_0 \in \mathbb{S}^k$. Finally, consider the family

$$\tilde{R} = (T_j \circ R_{s_0})^{-1}(T_j \circ R) : \mathbb{S}^k \rightarrow \mathcal{F}_{\Phi}^{-\infty}(X; E).$$

By construction, $\tilde{R}_{s_0} = \text{Id}$, and $\text{ind}(\tilde{R}) = [j] - [\mathcal{E}]$. Since any element of $\tilde{K}^0(\mathbb{S}^k)$ is of the form $[j] - [\mathcal{E}]$, $j = \text{rank } \mathcal{E}$, this shows that the index map is surjective. □

Theorem 7.19. *For $k \in \mathbb{N}_0$ and provided $\dim Z > 0$,*

$$\begin{aligned} \pi_{2k+1}(G_{\Phi}^{-\infty}(X; E)) &\cong \tilde{K}^{-1}(Y^{T^*Y}), \\ \pi_{2k}(G_{\Phi}^{-\infty}(X; E)) &\cong \ker [\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z}]. \end{aligned}$$

Proof. The case of the set of connected components was handled in lemma 7.11. For the remaining cases, the surjectivity of the boundary homomorphism allows us to decompose the long exact sequence of homotopy groups (7.2) into short exact sequences. More precisely, for $k \in \mathbb{N}$, we get the exact sequences

$$(7.10) \quad 0 \longrightarrow \pi_{2k-1}(G_{\Phi}^{-\infty}(X; E)) \longrightarrow \pi_{2k-1}(G_{\Phi s, 0}^{-\infty}(X; E)) \longrightarrow 0$$

and

$$(7.11) \quad 0 \longrightarrow \pi_{2k}(G_{\Phi}^{-\infty}(X; E)) \longrightarrow \pi_{2k}(G_{\Phi s, 0}^{-\infty}(X; E)) \xrightarrow{\partial} \pi_{2k-1}(\dot{G}^{-\infty}(X; E)) \longrightarrow 0,$$

where we used the fact that $\pi_{2k}(\dot{G}^{-\infty}(X; E)) \cong \{0\}$. For odd homotopy groups, the theorem easily follows from (7.10) and remark 7.8. For even homotopy groups,

notice that by proposition 7.15 and proposition 7.6, the exact sequence (7.11) can be rewritten

$$(7.12) \quad 0 \longrightarrow \pi_{2k}(G_{\Phi}^{-\infty}(X; E)) \longrightarrow \pi_{2k}(\mathcal{F}_{\Phi}^{-\infty}(X; E)) \xrightarrow{\text{ind}} \tilde{K}^0(\mathbb{S}^{2k}) \longrightarrow 0,$$

so $\pi_{2k}(G_{\Phi}^{-\infty}(X; E)) \cong \ker [\text{ind} : \pi_{2k}(\mathcal{F}_{\Phi}^{-\infty}(X; E)) \rightarrow \tilde{K}^0(\mathbb{S}^k)]$. But the homotopy group $\pi_{2k}(\mathcal{F}_{\Phi}^{-\infty}(X; E))$ is isomorphic to $K_c^0(T^*Y)$, which is a finitely generated \mathbb{Z} -module, and $\tilde{K}^0(\mathbb{S}^k) \cong \mathbb{Z}$. By the classification of finitely generated \mathbb{Z} -modules⁵, the isomorphism class of the kernel is the same for all surjective homomorphism $K_c^0(T^*Y) \rightarrow \mathbb{Z}$. Since $\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z}$ is also surjective, this means

$$\pi_{2k}(G_{\Phi}^{-\infty}(X; E)) \cong \ker [\text{ind}_t : K_c^0(T^*Y) \rightarrow \mathbb{Z}].$$

□

8. VANISHING OF THE HOMOTOPY GROUPS

In the case of cusp operators, that is, when the fibration $\Phi : \partial X \rightarrow \{\text{pt}\}$ is trivial, one gets as an easy corollary of theorem 7.19 the weak contractibility result of [17], namely that all the homotopy groups of $G_{\Phi}^{-\infty}(X; E)$ vanish.

Corollary 8.1. *If $\Phi : \partial X \rightarrow \{\text{pt}\}$ is the trivial fibration (cusp operators), then $G_{\Phi}^{-\infty}(X; E)$ is weakly contractible, meaning that all its homotopy groups are trivial, including the set of connected components.*

Proof. In this case, $Y = \text{pt}$, so T^*Y does not really make sense, but looking back at the identification (5.16) and the proof of lemma 7.5, we see that $K_c^0(T^*Y)$ and $\tilde{K}^{-1}(Y^{T^*Y})$ must be replaced by $\tilde{K}^0(\mathbb{S}^2) \cong \mathbb{Z}$ and $\tilde{K}^{-1}(\mathbb{S}^2) \cong \{0\}$ in the statement of theorem 7.19. The result then easily follows from the fact that $\text{ind}_t : \tilde{K}^0(\mathbb{S}^2) \rightarrow \mathbb{Z}$ is an isomorphism. □

In fact, it turns out that with only little more effort, one can deduce that $G_{\Phi}^{-\infty}(X; E)$ is actually contractible. It suffices to apply with slight modifications the argument of Kuiper in his proof of the contractibility of the group of invertible bounded operators acting on a separable Hilbert space (see [8]).

The main difference in our situation is that the topology we consider on the group $G_{\Phi}^{-\infty}(X; E)$ is not the one coming from the operator norm, but the \mathcal{C}^∞ -topology coming from the Schwartz kernels of the smoothing operators in $\Psi_{\Phi}^{-\infty}(X; E)$. Nevertheless, since $G_{\Phi}^{-\infty}(X; E) \subset \Psi_{\Phi}^0(X; E)$, we can also give to $G_{\Phi}^{-\infty}(X; E)$ the topology induced by the operator norm $\|\cdot\|$ of bounded operators acting on $L^2(X; E)$. This is a weaker topology than the \mathcal{C}^∞ -infinity topology, in the sense that it has less open sets. In the operator norm topology, the space $G_{\Phi}^{-\infty}(X; E)$ is easily seen to be a metric space. By a theorem of Stone⁶, metric spaces are paracompact. This will be very useful to retract $G_{\Phi}^{-\infty}(X; E)$ (in the \mathcal{C}^∞ -topology) onto a CW-complex.

Definition 8.2. *We will say that an open ball in $\text{Id} + \Psi_{\Phi}^{-\infty}(X; E)$ of radius ϵ*

$$B_{\epsilon}(A) = \{\text{Id} + Q \mid Q \in \Psi_{\Phi}^{-\infty}(X; E), \ \|A - \text{Id} - Q\| < \epsilon\}, \quad A \in G_{\Phi}^{-\infty}(X; E),$$

*is **small** if $B_{3\epsilon}(A) \subset G_{\Phi}^{-\infty}(X; E)$. Clearly, one can cover $G_{\Phi}^{-\infty}(X; E)$ by such balls.*

⁵See for instance in [6].

⁶See for instance theorem 4, section I.8.4, p.101 in [19]

Proposition 8.3. *If $G_{\Phi}^{-\infty}(X; E)$ is weakly contractible, then it is contractible. In particular, if $\Phi : \partial X \rightarrow \text{pt}$ is the trivial fibration, then $G_{\Phi}^{-\infty}(X; E)$ is contractible.*

Proof. Let $\{B_{\epsilon_i}(A_i)\}_{i \in I}$ be a covering of $G_{\Phi}^{-\infty}(X; E)$ by small balls,

$$(8.1) \quad G_{\Phi}^{-\infty}(X; E) = \bigcup_{i \in I} B_{\epsilon_i}(A_i).$$

Since the Banach space of bounded operators acting on $L^2(X; E)$ is separable, we can assume $I = \mathbb{N}$. Moreover, since $G_{\Phi}^{-\infty}(X; E)$ is paracompact in the operator norm topology, we can assume that the covering (8.1) is locally finite and that it has an associated partition of unity $\{\phi_i\}_{i \in \mathbb{N}}$. A priori, the ϕ_i are continuous with respect to the operator norm topology, but since the \mathcal{C}^∞ -topology is a finer topology, this means they are also continuous with respect to the \mathcal{C}^∞ -topology. For $t \in [0, 1]$, consider the following homotopy

$$(8.2) \quad \begin{aligned} \xi_t : G_{\Phi}^{-\infty}(X; E) &\rightarrow G_{\Phi}^{-\infty}(X; E) \\ z &\mapsto (1-t)z + t \sum_{i \in \mathbb{N}} \phi_i(z) A_i. \end{aligned}$$

For $t = 0$, ξ_0 is just the identity. To see that the image really lies in $G_{\Phi}^{-\infty}(X; E)$, let $z \in G_{\Phi}^{-\infty}(X; E)$ be given. Then there exists a neighborhood \mathcal{U} of z such that \mathcal{U} has a non-empty intersection with only finitely many open sets of the covering (8.1), say $B_{\epsilon_{i_1}}(A_{i_1}), \dots, B_{\epsilon_{i_m}}(A_{i_m})$. Without loss of generality, assume that $B_{\epsilon_{i_1}}(A_{i_1}), \dots, B_{\epsilon_{i_m}}(A_{i_m})$, $n \leq m$ are the open sets of the covering containing z , and assume that $\epsilon_{i_1} = \max\{\epsilon_{i_1}, \dots, \epsilon_{i_n}\}$. By construction, we have

$$(8.3) \quad z \in B_{\epsilon_{i_k}}(A_{i_k}) \subset B_{3\epsilon_{i_1}}(A_{i_1}), \quad \forall k \in \{1, \dots, n\},$$

which implies that $\xi_t(z) \in B_{3\epsilon_{i_1}}(A_{i_1}) \subset G_{\Phi}^{-\infty}(X; E)$ for all $t \in [0, 1]$.

Let N be the nerve of the covering (8.1). Call b_i the vertex that corresponds to the open set $B_{\epsilon_i}(A_i)$. Then N is a CW-complex with affine simplices as cells. There is an obvious inclusion $\rho : N \rightarrow G_{\Phi}^{-\infty}(X; E)$ given by sending the vertex b_i to A_i for $i \in \mathbb{N}$ and so that on any simplex of N , ρ is an affine map. Since $\xi_1(G_{\Phi}^{-\infty}(X; E)) \subset N$, the homotopy (8.2) shows that the inclusion ρ is a homotopy equivalence. By Whitehead's theorem, we then conclude that $G_{\Phi}^{-\infty}(X; E)$ is contractible. \square

REFERENCES

- [1] B. Ammann, R. Lauter, and V. Nistor, *Pseudodifferential operators on manifolds with a Lie structure at infinity*, math.AP/0304044.
- [2] M.F. Atiyah, *K-theory*, Benjamin, 1967.
- [3] M.F. Atiyah and I.M. Singer, *The index of elliptic operators: I*, Ann. of Math. **87** (1968), 484–530.
- [4] R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. **70** (1959), 313–337.
- [5] R. Bott and L.W. Tu, *Differential forms in algebraic topology*, no. 82, Springer-Verlag, Berlin, 1982.
- [6] B. Hartley and T.O. Hawkes, *Rings, modules and linear algebra*, Chapman and Hall, London, 1970.
- [7] L. Hörmander, *The analysis of linear partial differential operators. vol. 3*, Springer-Verlag, Berlin, 1985.
- [8] N.H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology **3** (1964), 19–30.
- [9] R. Lauter and S. Moroianu, *Fredholm theory for degenerate pseudodifferential operators on manifold with fibred boundaries*, Comm. Partial Differential Equations **26** (2001), 233–283.

- [10] ———, *Homology of pseudodifferential operators on manifolds with fibered cusps*, T. Am. Soc. **355** (2003), 3009–3046.
- [11] R. Mazzeo and R. B. Melrose, *Pseudodifferential operators on manifolds with fibred boundaries*, Asian J. Math. **2** (1999), no. 4, 833–866.
- [12] R.B. Melrose, *Analysis on manifolds with corners*, In preparation.
- [13] ———, *The Atiyah-Patodi-Singer index theorem*, A. K. Peters, Wellesley, Massachusetts, 1993.
- [14] ———, *The eta invariant and families of pseudodifferential operators*, Math. Res. Lett. **2** (1995), no. 5, 541–561. MR **96h**:58169
- [15] ———, *Geometric scattering theory*, Cambridge University Press, Cambridge, 1995.
- [16] R.B. Melrose and P. Piazza, *Families of Dirac operators, boundaries and the b-calculus*, J. Differential Geom. **46** (1997), no. 1, 99–180.
- [17] R.B. Melrose and F. Rochon, *Families index for pseudodifferential operators on manifolds with boundary*, IMRN **22** (2004), 1115–1141.
- [18] R. Nest and B. Tsygan, *Deformations of symplectic Lie algebroids, deformations of holomorphic symplectic structures, and index theorems*, Asian J. Math. **5** (2001), 599–635.
- [19] H. Shubert, *Topology*, Allyn and Bacon Inc, Boston, 1968.
- [20] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, New Jersey, 1999.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY
E-mail address: rochon@math.mit.edu